Math 414 Lecture 13

Finding an initial feasible basic solution

If the initial matrix has a set of m independent identity columns and the constant column b is ≥ 0, start with these. Otherwise make each constant column entry ≥ 0 by multiplying negative rows by -1.

Often a slack variable will have an identity column. Otherwise, it has a negated identity column or there is no slack variable (e.g. the constraint r is an equality). In either case, create an extra identity column by adding an *extra* variable r'. The added extra variables give the needed independent identity columns. Arrange the identity columns on the left so to form an identity matrix.

Phase one gets rid of these added variables by reducing them to 0. The phase-one objective is to minimize the sum of these variables, i.e., maximize e = the negation of this sum (in phase-one, the original objective is ignored). When this sum has been reduced to 0, proceed to phase two.

Phase two restores the original objective function and starts with the result of deleting the extra variables from the basic solution found at the end of phase one. The simplex algorithm is run as before. All phase one did was get an initial solution to start with.

Label the extra variables with a prime $'$. If constraint r needs both slack and extra variables, label the slack r and the extra variable r'.

The text calls extra variables "*artificial*" variables.

Two-phase method

Given a canonical problem with added slack variables:

- **Multiply any equations with negative constants by -1. This makes all the constants on the left ≥ 0.**
- **If the slack variable is either missing or negated, add a new *extra* variable to the equation.**
- **The result is the *phase-one* problem whose objective is to minimize (and thus get rid of) the sum of the extra variables.**

■ ORIGINAL PROBLEM

max $z = 5x - y$

r: $2x - y \geq 3$

s: $x + 9y = -2$

$x, y \geq 0$

CANONICAL WITH CONSTANTS ≥ 0

max $z = 5x - y$

r: $2x - y - r = 3$

s: $-x - 9y = 2$

$x, y, r \geq 0$

CANONICAL PLUS EXTRAS

max $z = 5x - y$

r: $2x - y - r + r' = 3$

s: $-x - 9y + s' = 2$

$x, y, r, r', s' \geq 0$

PHASE ONE PROBLEM

max $e = -r - s$

r: $2x - y - r + r' = 3$

s: $-x - 9y + s' = 2$

$x, y, r, r', s' \geq 0$

Solutions for the original problem correspond exactly to the phase-one solutions whose extra variables r', s' are 0. And these are the solutions for which $e = -r' - s' = 0$.

In the example, $[x, y]$ is a feasible solution to the original problem with slack r iff $[x, y, r, 0, 0]$ is a feasible solution to the extra problem iff $[x, y, r, r', s']$ is an optimal phase-one solution with $e = -r' - s' = 0$.

Clearly $[x, y, r, r', s'] = [0,0,0,3,2]$ is a feasible solution to the phase one problem. To convert this initial solution to a solution of the form $[x, y, r, 0, 0]$, we must reduce the extra variables to 0, i.e., we must maximize $e = -r' - s'$ to 0.

Warning, delete <u>only</u> the extra variables - they have primes $'$.

Phase one

Apply the simplex method to the phase-one problem,

- **Replace the objective with *max e* = -r'−s'.**
- **Form the initial augmented matrix for this problem. Put the original variable columns first. Arrange the identity columns at the end to form an identity matrix.** *Warning: this matrix is not a tableau.*
- **To get the first tableau, pivot on the identity columns.**
- **Run the simplex algorithm.**
- **If the optimal solution has a max value *e* = 0, begin phase two with this solution.**
- **If the phase-one optimal value is $\neq 0$, there is no feasible solution to the original problem. Thus the answer is** "*Empty region, no feasible solutions*".

Phase one matrix

	x	y	r	r'	s'	b	θ
	2	-1	-1	1	0	3	
	-1	-9	0	0	1	2	
e	0	0	0	1	1	0	

Tableau 1

	x	y	r	r'	s'	b	θ
r'	2	-1	-1	1	0	3	1.5
s'	-1	-9	0	0	1	2	
e	-1	10	1	0	0	-5	

Tableau 2

	x	y	r	r'	s'	b
x	1	-1/2	-1/2	0.5	0	3/2
s'	0	-19/2	-1/2	1/2	1	7/2
e	0	19/2	1/2	1/2	0	-7/2

The optimal solution has max $e = -r' - s' = -7/2 \neq 0$, thus the phase one problem has no solution whose extra variables are 0. Thus the original problem has no feasible solution and the answer is "*Empty region, no feasible solutions*".

Otherwise the first phase produces a solution with max $e = 0$. Thus the extra variables are 0. Proceed to phase two.

Phase two

If the extra variables of the phase-one solution are parameters, discarding them gives an initial basic feasible solution for the original problem. Run the simplex algorithm to get an an optimal answer.

If some extra variables are basic (and thus degenerate since they are 0), pivot on any nonzero entry of the extra variable's row. This makes a new entering basic variable and changes the departing unwanted extra variable into a parameter. Discard this extra parameter and run the simplex algorithm.