FROM LAST LECTURE.

**Primal problem**  CANONICAL+extras  **Dual**

\[
\begin{align*}
\text{max } z &= C \cdot X \\
AX &\leq B \\
X &\geq 0 \\
\end{align*}
\begin{align*}
\text{max } z &= C \cdot X \\
AX+... &= B \\
X, ..., &\geq 0 \\
\end{align*}
\begin{align*}
\text{min } z &= B \cdot W \\
A^T W &\geq C \\
W &? 0 \\
\end{align*}
\]

Where ? is some combination of \( \leq, \geq, = \).

The ... are the slack and extra variables.

Initially arrange the extras/slacks in an identity matrix \( T \).

The columns \( T \) are now retained (not deleted) after phase one.

But ignore the extra variables and their possibly negative objective coefficients when running the simplex algorithm.

The \( j \)th dual constraint will be

\[
a_{1j}w_2 + a_{2j}w_2 + \ldots + a_{mj}w_m \geq c_j
\]

Call the lefthand side \( z_j \).

**Lemma**

For each feasible (optimal / nonoptimal) solution \( X \) of the primal there is a solution \( W \) (feasible / nonfeasible resp.) of the dual with the same objective value, i.e., \( C \cdot X = z = B \cdot W \).
**Given.** A canonical tableau with extra variables for a primal problem whose initial constant column is \( B \).

Let \( X \) be the tableau’s solution.

Let \( W \) be the associated solution for the dual problem.

The cells of the initial identity matrix form a matrix \( T \).

Write the original objective function coefficients \( c_j \) (not their negatives) at the top of the tableau.

Label the rows with their basic variables.

To the left of a basic variable, write its objective coefficient.

Let \( C_B \) be this column of objective coefficients.

Let \( t_j \) be the \( j \)th column of the tableau.

Let \( z_j = C_B \cdot t_j \)

<table>
<thead>
<tr>
<th>( C_B = )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
<th>( c_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_3 )</td>
<td>( r )</td>
<td>-8</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( y )</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( z )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Objective Row Theorem**

(a) The $j$th entry in the objective row is $z_j - c_j = C_B \cdot t_j - c_j$.

(b) If the $j$th column is associated with a variable $x$ of the primal problem, then $z_j = \text{the left side of the dual constraint}$

$x : a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m \geq c_j$.

The $j$th objective row entry

$= z_j - c_j$

$= \text{the slack in the dual constraint for } x$.

<table>
<thead>
<tr>
<th></th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>0</th>
<th>0</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
<td>$r$</td>
<td>$w'$</td>
<td></td>
<td>$T* B =$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$r$</td>
<td>-8</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$y$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>$z$</td>
<td>$z_1 - c_1$</td>
<td>$z_2 - c_2$</td>
<td>$z_3$</td>
<td>$z_4$</td>
<td></td>
</tr>
</tbody>
</table>
**Objective Row Theorem - continued**

(c) If the $j$th column is associated with an initial extra or slack variable $w$ for a primal constraint then $c_j = 0$. Hence the objective row entry is $z_j - c_j = z_j - 0 = z_j$. For the dual variable $w$, either

$$w = z_j = C_B \cdot t_j = \text{the objective row entry}$$

or $w = -z_j$.

The sign is determined by the restriction $w \geq 0$, or $w \leq 0$. If $w$ is unrestricted, the Marginal Value Theorem determines the sign. Alternatively, you can just systematically keep track of sign changes: multiplying a constraint by -1 changes the sign of the dual variable.

<table>
<thead>
<tr>
<th>$C_B =$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>0</th>
<th>0</th>
<th>$T*B =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_3$</td>
<td>$r$</td>
<td>-8</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$y$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>$z$</td>
<td>$z_1 - c_1$</td>
<td>$z_2 - c_2$</td>
<td>$z_3$</td>
<td>$z_4$</td>
<td>$z_5$</td>
</tr>
</tbody>
</table>
**Definition.** A solution is *superoptimal* / *suboptimal* iff its objective value is better / worse than the optimal value. For max problems, better means $>;$ for min problems better means $<\text{ the optimal value}$.

Since optimal solutions are the best feasible solutions, superoptimal solutions which have better values can not be feasible.

Now we use the Objective Row Theorem to prove the Primal-Dual Theorem. Suppose the primal problem has standard form. Thus the dual variables are $\geq 0$. 
PRIMAL-DUAL THEOREM.

(A) For every feasible basic solution of a primal problem, there is an associated solution for the dual problem with the same objective value.

(B) The primal solution is optimal iff the objective row is $\geq 0$, iff the values of the dual solution and its slacks are $\geq 0$, iff the dual solution is feasible.

(C) The primal solution is suboptimal iff the dual solution is superoptimal iff the dual solution is not feasible.

(See picture on previous page.)
Recall that $B$ is the original constant column and $T$ is the matrix currently in the position of the initial identity matrix.

**Constant Column Theorem**

The current constant column = $T \cdot B$.

Proof. The reason this works is the same as the reason that transforming a matrix

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \text{ to } \begin{bmatrix} 1 & 0 & d & e \\ 0 & 1 & f & g \end{bmatrix}$$

via elementary row operations, produces the inverse $\begin{bmatrix} d & e \\ f & g \end{bmatrix}$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The identity matrix on the right side records the elementary row operations which transforms $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Multiplying $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $\begin{bmatrix} d & e \\ f & g \end{bmatrix}$ on the left plays back the these operations and produces $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. 
\begin{align*}
\text{max } z &= 2x + 3y. \text{ Initial solution } r = 10, s = 8, \text{ with } x, y \text{ parameters.}
\end{align*}

Initial tableau: Fill in the blanks.

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
 & $x$ & $y$ & $r$ & $s$ & $b$ \\
\hline
\hline
-- & -- & 1 & 0 & & \\
-- & -- & 0 & 1 & & \\
\hline
$z$ & & & & & \\
\hline
\end{tabular}
\end{table}

Subsequent tableau: Fill in the blanks.

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
 & $x$ & $y$ & $r$ & $s$ & $b$ \\
\hline
\hline
 & -8 & 0 & 1 & -1 & \\
3 & 1 & 0 & 1/2 & \\
\hline
$z$ & & & & & \\
\hline
\end{tabular}
\end{table}

\begin{align*}
\text{primal} & & \text{primal slacks} & & \text{dual} & & \text{dual slacks} \\
\\n$x =$ & & $r =$ & & $r =$ & & $x =$ \\
y =$ & & $s =$ & & $s =$ & & $y =$ \\
\end{align*}
\[
\text{max } z = \min z =
\]

- \( \text{max } z = 2x + y \).

Initial solution \( r = 0, s = 2, t = 3 \). \( x, y \) are parameters.

Fill in all blanks.

<table>
<thead>
<tr>
<th>primal</th>
<th>primal slacks</th>
<th>dual</th>
<th>dual slacks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = )</td>
<td>( r = )</td>
<td>( r = )</td>
<td>( x = )</td>
</tr>
<tr>
<td>( y = )</td>
<td>( s = )</td>
<td>( s = )</td>
<td>( y = )</td>
</tr>
<tr>
<td>( t = )</td>
<td>( t = )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\max z = \min z =
\]
Is the dual solution feasible?  yes  no

\[ \text{max } z = 2x + y. \]

Initial solution \( r=0, s=2, t=3, \) with \( x, y \) parameters.

Fill in all blanks.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
<td>r</td>
<td>s</td>
<td>t</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>( y )</td>
<td>0</td>
<td></td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>primal</th>
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<td>( x = )</td>
<td>( r = )</td>
<td>( r = )</td>
<td>( x = )</td>
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<tr>
<td>( y = )</td>
<td>( s = )</td>
<td>( s = )</td>
<td>( y = )</td>
</tr>
<tr>
<td>( t = )</td>
<td>( t = )</td>
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</tr>
</tbody>
</table>

\[ \text{max } z = \text{min } z = \]

Is the dual solution feasible?  yes  no  □
The simplex method chooses pivot points to guarantee that the next basic solution is feasible and has a better objective value.

But if you only wish to maintain feasibility, you may ignore the rule for choosing the column with the most negative objective coefficient. This rule insures chooses a direction leading to an improved objective value. Ignoring this rule may worsen the objective value but does not lead to infeasibility.

**Lemma.** For any entry in a tableau for a feasible basic solution, if the $b$ in the constant column is 0 and the entry coefficient is $\neq 0$, one can pivot on the entry and retain feasibility.
Examples of pivots which do and don’t maintain feasibility. The constant column is shown but not the objective row.

<table>
<thead>
<tr>
<th>Before pivot</th>
<th>After pivot</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( u )</td>
</tr>
<tr>
<td>( y )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 2 )</td>
</tr>
</tbody>
</table>
Examples of pivots which do and don’t maintain feasibility. The constant column is shown but not the objective row.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>-1</td>
<td>1</td>
<td>$\theta$</td>
</tr>
<tr>
<td>$y$</td>
<td>1</td>
<td>2</td>
<td></td>
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</tbody>
</table>

Before pivot

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
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<td>$\theta$</td>
</tr>
<tr>
<td>$y$</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

After pivot

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td></td>
<td></td>
<td>$\theta$</td>
</tr>
<tr>
<td>$y$</td>
<td></td>
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</tbody>
</table>

Can’t pivot on 0.
**Phase two case with a basic extra variable**

**Recall:** You may delete phase one columns which belong to extra variables which are parameters.

If an extra variable is basic, ignore the “pick most negative objective coefficient” rule. Instead pivot on any other nonzero coefficient in the extra variable's row. After pivoting, the extra basic variable departs and you may delete its column.

If you need the dual solution, keep the extra variable columns but **don’t pivot on their entries. Ignore their objective values even if negative.** The objective row contains the dual variable values up to ±. The signs are determined separately.

Previously, we pivoted on the initial phase 1 and phase 2 matrices to get the initial tableaus. These pivots changed only the objective row. Now, instead of pivoting, you can use the Objective Row Theorem to get the initial objective row directly.
FACTORS WHICH CHANGE A DUAL VARIABLE’S SIGN:

- Changing max to min.
- Multiplying a primal constraint by -1.
- Subtracting a slack instead of adding.