# Math 414 Lecture 14 Duality

For a variable $x$ in a general linear programming problem, we may have: $x \ge 0$, $x \le 0$, or $x$ unrestricted.

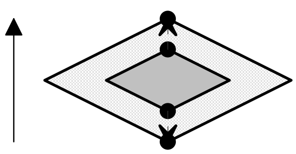
Suppose we increase or decrease the constant of a constraint. The constraint is *loosened* if it becomes easier to satisfy. It is *tightened* if it becomes harder to satisfy.

■ In $2x - y \le 8$, increasing the constant 8 loosens the constraint, decreasing the constant tightens it.

In $2x - y \ge 5$, increasing 5 tightens? loosens?
decreasing 5 tightens? loosens?

An optimal value *improves* if it is a higher max or a lower min. An optimal value may improve, remain constant, or *worsen*.

LEMMA.
Tightening/loosening an inequality constraint worsens/improves the optimal value, if it affects it at all.



Loosening a constraint, enlarges the region, increases (improves) the max, decreases (improves) the min.

A constraint is *upward* if increasing its constant increases the optimal value. It is *downward* if increasing the constant decreases the optimal value.

For max problems, $\le$ is upward.
For min problems, $\ge$ is upward.

■ max $z = x + y$
$r$: $3x + y \le 8$ ← upward, increasing 8 loosens constraint, raises the max
$s$: $2x + y \ge 4$ ← downward, increasing 4 tightens constraint, lowers max.
$x, y \ge 0$

■ min $z = x + y$
$r$: $3x + y \le 8$ ← upward? downward?
$s$: $2x + y \ge 4$ ← upward? downward?
$x, y \ge 0$

DEFINITION OF DUAL

| PRIMAL | | DUAL | |
|---|---|---|---|
| max/min | $z = C \bullet X$ | min/max | $z = B \bullet W$ |
| $W$: $A \bullet X$ ? $B$ | X ? 0 | $X$: $A^T \bullet W$ ? $C$ | W ? 0 |

Suppose the variables are $X$, the constraints are labeled with variables $W$, the objective vector is $C$ (thus $z = C \bullet X$), the coefficient matrix is $A$ and the constant column is $B$, then in the *dual problem*:

► Max and min are exchanged.

► Variables $X$ and constraint labels $W$ are exchanged.

► The constant column $B$ and the objective vector $C$ are exchanged. In the dual problem, $C$ is the constant column and $z = B \bullet W$ is the objective function.

► The coefficient matrix is transposed, $A \to A^T$.

► Upward constraints ↔ dual variables which are $\ge 0$.
Downward constraints ↔ dual variables $\le 0$.
Equalities ↔ unrestricted variables.

The original problem is called the *primal problem*.

For any constraint $s$, the value of its *dual variable s* = the rate of change of the optimal value with respect to the constant of the constraint. Put simply

MARGINAL VALUE THEOREM. If $s$ is the dual variable of a constraint, then adding $\pm 1$ to the constant increases (other constraints permitting) the optimal value $z$ by $\pm s$.

The Marginal Value Theorem also determines the dual units. If the optimal value $z$ is dollars and the constraint constant $b$ is hours, then the dual variable $s$ is the rate of change of $z$ w.r.t $b = \Delta z / \Delta b$. The units are dollars/hour.

Basically, $s$ = the amount the optimal value increases when the constant in the constraint is increased by 1.

In economics, the dual variables are known as *shadow prices* (Linsolve's word) or *marginal values*.

If you multiply a constraint by -1, you have to replace the dual variable $s$ by $-s$. Unlike two-phase, for duality, we usually avoid multiplying by -1.

■ Find the duals of the problems below.

| PRIMAL PROBLEM | DUAL PROBLEM |
|---|---|
| Max $z = C \bullet X$ | Min $z = B \bullet W$ |
| with | with |
| $W$: $AX \le B$ | $X$: $A^T W \ge C$ |
| $X \ge 0$ | $W \ge 0$. |

| PRIMAL | DUAL |
|---|---|
| Max $z = C \bullet X$ | Min $z = B \bullet W$ |
| with | with |
| $W$: $AX = B$ | $X$: $A^T W \ge C$ |
| $X \ge 0$ | $W$ unrestricted. |

| PRIMAL | DUAL |
|---|---|
| Max $z = 8x + 9y$ | Min $z = 5r + 6s$ |
| with $x$ $y$ | with $r$ $s$ |
| $r$: $2x - 3y \le 5$ | $x$: $2r + 4s \ge 8$ |
| $s$: $4x + 8y \ge 6$ | $y$: $-3r + 8s = 9$ |
| $x \ge 0$, $y$ unrestricted | $r \ge 0$, $s \le 0$. |

| PRIMAL | DUAL |
|---|---|
| Max $z = 8x + 9y$ | Min $z = 5r + 6s$ |
| with $x$ $y$ | with $r$ $s$ |
| $r$: $2x - 3y = 5$ | $x$: $2r + 4s \ge 8$ |
| $s$: $4x + 8y \ge 6$ | $y$: $-3r + 8s \ge 9$ |
| $x, y \ge 0$ | $r$ unrestricted, $s \le 0$. |

| PRIMAL | DUAL |
|---|---|
| Min $z = 8x + 9y$ | Max $z = 5r + 6s$ |
| with $x$ $y$ | with $r$ $s$ |
| $r$: $2x - 3y \ge 5$ | $x$: $2r + 4s \le 8$ |
| $s$: $4x + 8y \le 6$ | $y$: $-3r + 8s = 9$ |
| $x \ge 0$, $y$ unrestricted | $r \ge 0$, $s \le 0$. |