Math 414  Lecture 15

One step of the Simplex method makes feasible (constant-column is \( \geq 0 \)) but nonoptimal solutions (objective row has a negative coefficient) more optimal.

The dual method makes optimal (or super-optimal) (objective row coefficients are \( \geq 0 \)) but nonfeasible solutions (constant column has a negative) more feasible. The algorithm is dual to the simplex method. It reverses: rows/columns, negative/positive \( q \)-ratio coefficients, departing/entering.

**Dual method**

Input: a tableau for a super-optimal solution (objective row coefficients are \( >0 \), some constant column entry is \( <0 \)).

Loop:
- Select the row with the most negative constant-column entry. Its variable will depart.
- If the row has no negative coefficients, stop, there is no feasible solution. Otherwise, select the negative coefficient, whose objective/(negative-coefficient) ratio is nearest to 0. Its variable enters.
- Pivot on the selected row and column.
- If all the constant-column entries are \( >0 \), stop, a feasible optimal solution has been found.
- Otherwise repeat the loop.

After adding a new constraint, optimal solutions retain optimality; (they are remain optimal or become super-optimal). If the solution fails to satisfy the new constraint, it becomes nonfeasible with the constraint’s slack being negative. In this case, apply the dual method to restore feasibility.

**Given problem.**

\[
\begin{align*}
\text{max } & z = y \\
\text{r: } & x + y \leq 2 \\
\end{align*}
\]

Adding the constraint \( p: -x + y \geq 0 \) eliminates the dark region. The original optimal solution remains optimal and feasible.

Adding the constraint \( q: -x + y \leq 0 \) shrinks the region to the dark region. The original optimal solution becomes super-optimal and nonfeasible.

The optimal solution for the original problem is: \( \max z=2, \text{ at } x=0, y=2 \text{ with slack } r=0 \).

Adding constraint \( q \) makes this super-optimal and nonfeasible and the new optimal solution is: \( \max z=1, \text{ when } x=1, y=1, \text{ with slacks } r=0, q=0 \).

The dual method works backward from the original solution (which is now nonfeasible and superoptimal) to get this new feasible optimal solution.

The final tableau for the original problem is:

\[
\begin{array}{cccccc}
 x & y & r & b \\
 \hline
 y & 1 & 1 & 1 & 2 \\
 z & 1 & 0 & 1 & 2 \\
\end{array}
\]

- Make the new constraint \( a \leq \) inequality.
  - If it is an \( = \), rewrite it as two \( < \)'s.
  - If it is a \( > \) inequality, multiply by -1 (even if the constant becomes negative) to make it a \( \leq \). This guarantees there is a slack but no extra variable.

  - Initial dual-method matrix: Add the new constraint just above the objective row and add a column for the new slack variable. Warning, after adding the new row, the resulting matrix is usually not a tableau. Correct this by pivoting.
  - Initial tableau: Pivot on the 1’s in the basic columns, \( y \) in this case, to restore them to identity columns.

  **Initial matrix** with new constraint and slack variable.

\[
\begin{array}{cccccc}
 x & y & r & q & b \\
 \hline
 y & 1 & 1 & 1 & 0 & 2 \\
 q & -1 & 1 & 0 & 1 & 0 \\
 z & 1 & 0 & 1 & 0 & 2 \\
\end{array}
\]

  **Initial tableau**

\[
\begin{array}{cccccc}
 x & y & r & q & b \\
 \hline
 y & 1 & 1 & 1 & 0 & 2 \\
 q & -2 & 0 & -1 & 1 & -2 \\
 z & 1 & 0 & 1 & 0 & 2 \\
 0 & -\frac{1}{2} & -1 & & & \\
\end{array}
\]

In the constant-column, the \( q \) row is \( <0 \), so \( q \) departs.
In \( q \)’s coefficient row \( x, r \) have negative coefficients, and ratios \(-\frac{1}{2}, -1, -\frac{1}{2}\) is nearest 0; \( x \) enters. Pivoting gives.

\[
\begin{array}{cccccc}
 x & y & r & q & b \\
 \hline
 y & 0 & 1 & 1/2 & 1/2 & 1 \\
 x & 1 & 0 & 1/2 & -1/2 & 1 \\
 z & 0 & 0 & 1/2 & 1/2 & 1 \\
\end{array}
\]

Since the constant column is \( \geq \) we have found a feasible optimal solution.

**Answer:** \( \max z = 1 \text{ at } x = 1, y = 1 \text{ with slacks } r = 0, q = 0 \).
Recall: **Constant Column Theorem.** Place the initial basic identity columns at the end to make an identity matrix.

Let $T$ = the matrix in that region of the final tableau.
Let $B = \text{original constant \ column (exclude the objective value).}$ Assume the constant column entries are $\geq 0$.
Let $B' = \text{the final constant \ column, then}$
\[ T \cdot B = B' \cdot \]

**Initial matrix**
\[
\begin{array}{cccccc}
\text{ } & x & y & s' & r & s & b \\
\hline
r & 1 & -2 & 0 & 1 & 0 & \text{10} \\
s & 1 & 1 & -1 & 0 & 1 & \text{5} \\
z & - & - & - & - & - & - \\
\end{array}
\]

Complete the final tableau
\[
\begin{array}{cccccc}
\text{ } & x & y & s' & r & s & b' \\
\hline
r & 3 & 0 & -2 & 1 & 2 & \text{ } \\
y & 1 & 1 & -1 & 0 & 1 & \text{ } \\
z & - & - & - & - & - & - \\
\end{array}
\]

Original $B = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, final $B' = T \cdot B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$.

**Constant column sensitivity**
Identify a “solution” with the constraint boundaries on which it lies (the constraints with 0 slacks). Thus a solution will move with its constraint lines.
Let $b_r$ = the right-hand constant of constraint $r$.
When this constant changes, the constraint’s boundary line moves. When the boundary line moves, so do the extremes on the boundary. So does any optimal extreme on the boundary. As it moves, it remains optimal. But if it is moved too far, it may become infeasible. We want the interval over which $b_r$ can range without the optimal solution becoming infeasible.

**Primal problem**
max $z = y$
with
\[ r: -x + y \leq 1 \leftarrow b_r \\
s: x + y \leq 3 \leftarrow b_s \\
t: x - y \leq 1 \leftarrow b_t \\
x, y \geq 0 \]
What happens when $b_s=3$ changes?
What is the range of values for $b_s$ for which the optimal solution remains feasible?

What happens when $b_r=1$ changes?
What is the interval of values for $b_r$ for which the optimal solution remains feasible?

What happens when $b_s=1$ changes? What is the interval of values for $b_s$ for which the optimal solution remains feasible?

What happens when $b_t=1$ changes? What is the interval of values for $b_t$ for which the optimal solution remains feasible?