Math 414 Lecture 15 Exam 2 next week

Recall:

Upward constraints ↔ dual variables ≥ 0,
Downward constraints ↔ dual variables ≤ 0,
equalities ↔ unrestricted dual variables.

■ PRIMAL | DUAL

PRIMAL	DUAL
min $z = x+y$	max $z =$
r: $-x+y \le 1$	x:
s: $x+y \ge 3$	y:
t: $x-y \le 1$	
$x, y \ge 0$	r s t

Recall: For a primal constraint s, the dual variable s =, other constraints permitting, the amount the optimal value increases when the constraint constant increases by 1.

■ Solve the above problem geometrically and calculate the dual variables as rates of change.

Recall: Equality constraints correspond to unrestricted dual variables.

PRIMAL	DUAL
max $z = 7x+8y$	
r: $2x-y \le 3$	
s: $x+y = 4$	
$x, y \ge 0$	

Primal and dual problems have the same optimal value.

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PRIMAL	DUAL
max $z = x$	min $z' = w$
w: $x \le 1$	x: $w \ge 1$
$x \ge 0$	$w \ge 0$
ANS. $z = 1$ at $x = 1$	ANS. $z' = 1$ at $w = 1$

Note: the optimal values z and z' are the same.

■

PRIMAL	DUAL
max $z = 5x$	min $z' = 3w$
w: $x \le 3$	x: $w \ge 5$
$x \ge 0$	$w \ge 0$
ANS. max $z = 15$ at $x = 3$	ANS. min $z' = 15$ at $w = 5$

For any feasible solution x of this primal problem and any feasible solution w of the dual problem we have

Since $x \le 3$ $5 \le w$.

$$z = 5x \le 5\cdot 3 = 15 = 3\cdot 5 \le 3w = z'.$$

Thus $z \le z'$.

In general, the optimal values z and z' of a primal and a dual problem are the same.

For nonoptimal values,
the objective z of the maximizing problem is
≤ the objective z' of the minimizing problem.

DUALITY THEOREM (Gale, Kuhn, Tucker). For any primal problem and its dual:

(a) An optimal value for one problem is also an optimal value for the other. (In economics, maximizing profits and minimizing costs are dual but equivalent objectives.)

(b) The objective values of feasible solutions of the maximizing problem are ≤ those of the minimizing problem.

(c) If both problems have feasible solutions, then both problems have optimal solutions.

(d) If a feasible primal solution has the same objective value as a feasible dual solution then both are optimal.

(e) One has no feasible solutions iff the other is unbounded in the gradient direction.

PROOF OF (a) Roughly speaking (there are sign changes and missing/added variables), an optimal tableau for the primal problem transposes to an optimal tableau for the dual problem. Since the optimal value lies in the bottom right corner, it transposes to itself. Hence it is the same for both problems.

PROOF OF (b) Suppose the optimal objective value for both problems is z. Let z_x be some nonoptimal value for the maximizing problem; let z_n be some nonoptimal value for the minimizing problem. Then $z_x \le z$ since z is the maximum value for the maximizing problem and $z \le z_n$ since z is the minimum value for the minimizing problem. Thus $z_x \le z \le z_n$ which implies $z_x \le z_n$.

PROOF OF (c). If the minimizing problem has a feasible solution, then its objective value is ≥ that of all maximizing solutions. Hence the nonempty set of maximizing solutions is bounded in the gradient direction and hence has an optimum value.

PROOF OF (d). Suppose both maximizing and minimizing problems have a common objective value z. By part (b) this value is ≥ to all values of the maximizing problem. Hence it is the optimal value for the maximizing problem. Similarly it is optimal value for the minimizing problem.

PROOF OF (e). Draw a picture. □

Henceforth, we will use the same optimal variable, say z, for both primal and dual problems.

LinSolve doesn't do negative variables. For these, you have to take care of the sign. To do $x \le 0$, replace x with $-x$ (thus $x \ge 0$). Run LinSolve. Negate LinSolve's answer to get back x.

Multiplying a constraint by -1 changes the sign of the dual variable for that constraint. Compare:

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PRIMAL	NEGATED PRIMAL
max $z = 5x$	max $z = 5x$
w: $x \le 3$	$-w$: $-x \ge -3$
$x \ge 0$	$x \ge 0$
DUAL $w \ge 0$	DUAL $-w \le 0$