Math 414 Lecture 16 Exam 2 includes this lecture.

DOUBLE DUAL THEOREM. The dual of a dual is the original primal problem.

PROOF FOR STANDARD CASE.

PRIMAL	DUAL	DUAL OF DUAL
max $z = C \bullet X$	min $z = B \bullet W$	max $z = C \bullet X$
with	with	with
W: $AX \le B$	X: $A^T W \ge C$	W: $A^{TT} X \le B$
$X \ge 0$	$W \ge 0$	$X \ge 0$

Transposing twice, $A^{TT} = A$, gives the original matrix.

COMPLEMENTARY SLACKNESS THEOREM. For any constraint with a slack variable and a dual variable: At least one of of two variables is 0. In fact, at least one is a parameter.

PROOF. The slack measures the distance between the basic solution and the constraint boundary.

The dual variable is the rate of change of the optimal value w.r.t. the constraint constant.

Changing the constraint constant moves the boundary for the constraint.

Assume the dual variable is nonzero.

∴ the optimal value changes when the constant changes.
∴ the optimal value changes when boundary moves.
∴ the optimal solution moves when the boundary moves.
∴ the optimal solution is on the boundary.
∴ the slack is zero.

Assume the slack is nonzero.

∴ the optimal solution is not on the constraint boundary.
∴ the optimal solution doesn't move when the boundary moves.
∴ the optimal value doesn't change when the constraint constant changes.
∴ the rate of change of the optimal value w.r.t the constraint constant is 0.
∴ the dual variable is 0.

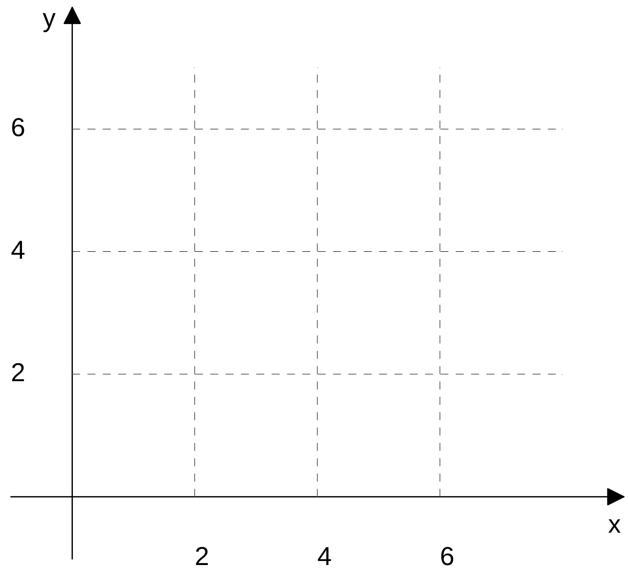
■ Calculate the slacks and dual variables geometrically.

Label the lines r, $r+1$, s, $s+1$, t, $t+1$.

PRIMAL PROBLEM	CANONICAL	DUAL
max $z = x + y$	max $z = x + y$	min $z = 4r + 6s + 4t$
with	with	with
r: $y \le 4$		
s: $x + y \le 6$		
t: $x \le 4$		
$x, y \ge 0$		

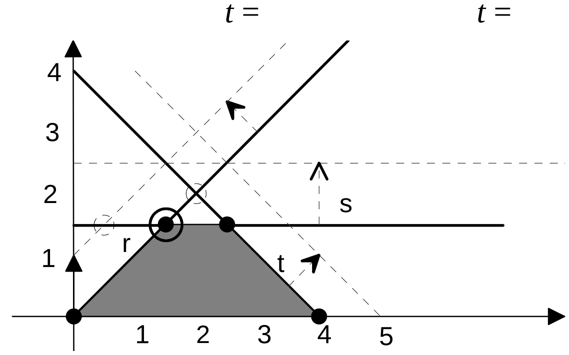
OPTIMAL EXTREME (if more than one, pick one)

max z = when	with slacks	and duals
x =	r =	r =
y =	s =	s =
	t =	t =



Recall: MARGINAL VALUE THEOREM. If s is the dual variable of a constraint, then adding ± 1 to the constraint constant increases (other constraints permitting) the optimal value by $\pm s$. If other constraints don't permit an increase as large as 1, increase by a smaller amount Δb. If Δz is the change in the objective value, then s = the rate of change of z w.r.t. $b = \Delta z / \Delta b$.

■ PRIMAL PROBLEM	DUAL PROBLEM	
max $z = y$	min $z = 0r + 1.5s + 4t$	
with	with	
r: $-x + y \le 0$	x: $r + t \ge 0$	
s: $y \le 1.5$	y: $s + r + t \ge 1$	
t: $x + y \le 4$	$s, r, t \ge 0$	
$x, y \ge 0$		
max z=	min z=	primal
when	when	slacks
$x = 1.5$	s =	s =
$y = 1.5$	r =	r =
	t =	t =



In constraint s, increasing 1.5 by 1, gives 2.5. But this goes too far, putting the constraint outside the feasible region. You can increase it by .5 and the max goes from 1.5 to 2 giving an increase of .5.

The dual variable $s = \Delta z / \Delta b = .5 / .5 = 1$.

The slack is $s = 0$.