

Math 414 Lecture 17 Use Hw17 to practice for exam.

PROOF OF THE MARGINAL VALUE THEOREM FROM THE DUALITY THEOREM: A constraint's dual variable is the rate of change of the optimal value w.r.t. the constraint's constant.

PRIMAL	DUAL
$\max z = c_1x + c_2y$	$\min z = b_1s + b_2t$
$s: a_1x + a_2y \leq b_1$	$x: a_1s + a_3t \geq c_1$
$t: a_3x + a_4y \leq b_2$	$y: a_2s + a_4t \geq c_2$

For constraint s , suppose we increase b_1 to $b_1 + \Delta b_1$.

Then $\min z = b_1s + b_2t$ increases to

$$\min z = (b_1 + \Delta b_1)s + b_2t = (b_1s + b_2t) + \Delta b_1s \text{ and } \Delta z = \Delta b_1s$$

Thus the rate of change $\Delta z / \Delta b_1 = \Delta b_1s / \Delta b_1 = s$.

By the Duality Theorem, $\max z$ of the Primal = $\min z$ of the Dual.

Hence the rate of change of optimal value of the primal problem is s . This is the Marginal Value Theorem.

This also determines the dual units. If z is dollars and b is hours, then the dual variable s is the rate of change of z w.r.t $b = \Delta z / \Delta b$. The units are dollars/hour. We will use this fact in the work problems below.

Duality in economics

Sawmill problem (see text).

You saw and plane rough and fine lumber.

For 1000' of rough lumber: saw 2 hours & plane 3 hours.

For 1000' of fine lumber: saw 2 hours and plane 5 hours.

The saw is available 8 hours (per day).

The plane is available 15 hours (per day).

The profit on 1000' of rough lumber = \$100.

The profit on 1000' of fine lumber = \$120.

Maximize the profit generated each day.

Let

r = thousands of feet of rough lumber produced per day.

f = thousands of feet of fine lumber produced per day.

P = total profit generated per day.

<i>Max profit problem</i>	<i>Minimum rent problem</i>
rough fine	saw plane
$\max \$ P = 100r + 120f$	$\min \$ R = 8s + 15p$
with	with
saw $s: 2r + 2f \leq 8$ hrs	rough $r: 2s + 3p \geq \$100$
plane $p: 3r + 5f \leq 15$ hrs	fine $f: 2s + 5p \geq \$120$
$r, f \geq 0$	$s, p \geq 0$

The min rent problem is dual to the max profit problem.

The units for r and f are feet, actually thousands of feet.

Units for dual variables:

s, p are $\Delta z / \Delta b =$ dollars / hour

- A stranger wants to rent your machines for a day - the saw for 8 hours, the plane for 15 hours. What is the minimum rent you are willing to accept?

Let

s = rental rate for the saw in dollars/hour.

p = rental rate for the plane dollars/hour.

R = one day's total rent.

Find the constraints for acceptable rental rates s and p .

$$\min \$ R = 8s + 15p$$

with

$$\text{rough } r: 2s + 3p \geq \$100$$

$$\text{fine } f: 2s + 5p \geq \$120 \quad s, p \geq 0$$

By the Duality Theorem, $P \leq R$ and $\max P = \min R$. This is obvious economically. You should rent out the machine only if the amount of rent exceeds the profit the machine makes. The minimum acceptable rent = the maximum possible profit.

The optimal solutions for the problems are:

Max profit for one day =

$$\max P = \$430 \text{ when } r = 2.5 \text{ (2500 ft), } f = 1.5.$$

Minimum rental for one day =

$$\min R = \$430 \text{ when } s = \$35/\text{hr}, p = \$10/\text{hr}.$$

- How much profit does the saw generate in an hour?

The amount of profit it generates = the amount the mills max profit P increases when the saw is run an additional hour, i.e., if the 8 hour constraint is increased to 9 hours. By the Marginal Value Theorem, this increase = the value of the dual variable s

= the minimum acceptable rental value of the saw = \$35.

This amount of profit the saw generates is also called the saw's shadow or accounting price or marginal value. The minimum acceptable saw rental rate is the *economic interpretation* of s , the additional profit generated when the saw is used an additional hour is the *marginal-value interpretation*.

s is usually \geq than the actual cost of running the saw.

- Which is more valuable? the saw or the plane?

The amount of profit / hour the saw generates = $s =$ \$35/hour. The marginal value of the plane = $p =$ \$10/hour. Thus the saw is more valuable.

- A worker wants an additional hour. Should you assign him to the saw or the plane?

Assign him to whichever generates the greater profit per hour. The saw generates $s =$ \$35/hour, the plane generates $p =$ \$10/hour. Hence the added time should be given to sawing.

- You must reduce the total workload by an hour. Take an hour away from the saw or the plane?

Reduce the one which will cause the least loss of profit, i.e., reduce the time used for planing.