Math 414 Lecture 23

Maximum Flows

- If the amount of water flowing into a node $x$ increases by 5, how much can the flow into adjacent nodes increase? The upper number = the capacity; the lower = the flow.

We are asking for potential individual increases; these increases are not simultaneously achievable.

Remaining capacity = edge capacity - edge flow.

**Potential increase calculation procedure**

Suppose $x$ is labeled with a potential increase in the amount of water available to it. Suppose $y$ is an unlabeled node connected (in either direction) to $x$.

The potential increase at $y$ is:

- $x \rightarrow y$ edges: min(potential increase at $x$, remaining capacity).
- $y \rightarrow x$ edges: min(potential increase at $x$, flow on the edge).

Label $y$ with this increase if it is $> 0$. Otherwise, leave $y$ unlabeled. Mark the edge with a check $✓$.

**Max-flow algorithm.** Don’t have to state it; be able to run it.

**Input:** A network.

**Output:** A maximal flow and a minimal cut.

**Comment:** At each step we label nodes with potential increases in flow. Then we either label edges with a new increased flow or make a minimum capacity cut.

- Start with 0 flow along each edge.
- A: (We look for a path along which the current flow can increase.)
  - Label the source with $\infty$.
  - B: Pick an edge between (in either direction) a labeled and an unlabeled node with a positive potential increase. Label the unlabeled node with this potential increase. Put a check on the edge.
    - If the sink is unlabeled and more nodes can be labeled, return to B.
    - If the sink is labeled with an increase $d$, work backward to the source to get a path along the checked edges (forward or backward). For each forward edge along the path, increase the flow by $d$; for each backward edge, decrease its flow by $d$.
      - Record the new flow (both the changed flows along the path and the unchanged flows not on the path).
      - Remove all node labels and edge checks. Go to A.

**Marriage problem.** There are 4 boys $\{A, B, C, D\}$ and 5 girls $\{M, N, O, P, Q\}$. “×” marks couples who will dance together. Find a matching with the most dance couples.

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The shortest path problem

Given: a connected undirected weighted graph with an **origin** node. Here, the weight of an edge is its length. Goal: find the shortest path from each node to the origin. The distance between two nodes or between a node and an edge is the length of the shortest path which contains both of them.

- In the graph below, the distance from b to a is 3+1+2 = 6. The distance between edge E and a is also 3+1+2 = 6.

![Graph Diagram]

Note that the length of an edge is included in calculating the distance of the edge to a node.

- Three edges leave the origin as shown below. Find the distances to the first three edges. Which node has a known distance? Which node distances are unknown? If the three nodes shown are the closer to the origin than the other nodes, what are their distances?

![Graph Diagram]

When a node’s distance to the origin has been calculated, it becomes a known node. An edge between a known node and an unknown node is a cut edge. Its estimated distance is its known node’s distance plus its length. The cut edges form a cut between the known nodes and the unknown nodes. Every path from the origin an unknown node must cross one of these cut edges.

- All cut edges are shown.

Fill their boxes with their estimated distances.

Find the correct distance for one of the unknown nodes.

**Lemma.**
- The distance from the origin to another node = the minimum of the distances to its edges.
- If the estimated distance of a cut edge is ≤ the estimated distances of other cut edges, then the estimated distance is the true distance between the edge and the origin.