Math 414 Lecture 26

The Cutting Plane Method

DEFINITION. For any real r,

- $[r]$ = the *floor of* r = the closest integer $\le r$.
- The number r is *integral* iff $r \in Z = \{..., -2,-1,0,1,2, ...\}$.

■ $[3.66]=3$, .66 is the *decimal part* of 3.66.
$[3.00]=3$, $[-3]=-3$, $[-3.3]=-4$ Not -3.

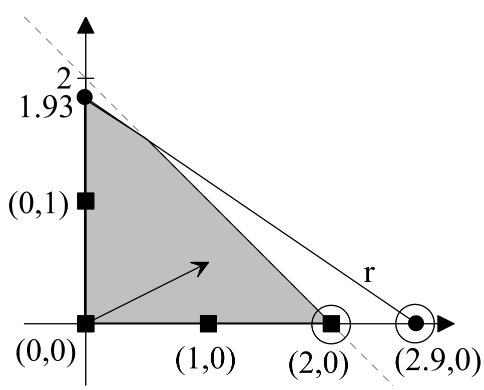
FACTS.

- $r-1<[r]\le r$.
- $[n]=n$ if $n\in Z$.
- For $n\in Z$, $n\le r \Rightarrow n\le [r]$.

Here's a problem in standard form (with ≤ not = or ≥).

■ max $z = 2x + y$
with
a: $x+(1.5)y \le (2.9)$
$x,y\in N$

Optimal noninteger solution:
$x=2.9$, $y=0$, $z=5.8$.
Feasible integer solutions (x, y, z):
$(0,0,0)$, $(0,1,1)$, $(1,0,2)$, $(1,1,3)$, $(2,0,4)$.
Optimal integer solution:
$x=2$, $y=0$, $z=4$.



We want an additional constraint which
(a) cuts off the current optimal noninteger solution but
(b) doesn't cut off any integer solution.

original constraint: $(1)x+(1.5)y \le (2.9)$
added new constraint: $[1]x+[1.5]y \le [2.9]$
which simplifies to $x+y\le 2$ -the dotted line.

With the added new constraint
(a) The old nonintegral optimal solution (2.9, 0) is cut off.
(b) No integral solution is cutoff . For any integral solution (x, y):

$(1)x+(1.5)y \le (2.9)$ implies, for any $x, y \ge 0$,
$[1]x+[1.5]y \le (2.9)$ implies, for x, y integral,
$[1]x+[1.5]y \le [2.9]$ (see the third fact above).

Stated directly without the [] symbols, the argument is:
$(1)x+(1.5)y \le (2.9)$ implies, for any $x, y \ge 0$,
$1x + 1y \le (2.9)$ implies, for x, y integral,
$1x + 1y \le 2$

In order to assure that the slacks are also integral, clear the fractions in the original constraints. Hence
$(1)x+(1.5)y \le (2.9)$ would be multiplied by 10 to get
$10x+ 15y \le 29$
This doesn't change the values of x or y but does make the slacks integral when x and y are integral.

CUTTING PLANE METHOD. Convert the problem to standard form. If the current primal variables are not integral, add a new integer-coefficient constraint to cut it off. When cut off, it becomes nonfeasible. Use the dual method to get back to a feasible solution. Repeat until an optimal integer solution is found.

THE CUTTING PLANE ALGORITHM FOR INTEGER PROBLEMS

- Run the simplex method as usual and get a solution.
  Loop:
- If the primal variables are integral, stop. We're done.
- If not pick the constraint for the primal variable (don't try to make slacks integral) with the largest decimal part (choose either one if two or more have the same largest decimal part).
- Take the floor of its coefficients and its constant, and add a slack variable.
- Add this new constraint, pivot to make a tableau whose basic variables have identity matrix columns.
- Apply the dual method to restore feasiblity and get a new solution.
  Repeat the loop.

For integer problems duality theory fails. Omit dual variable calculations.

MatLab example. Let's add a constraint which cuts of the nonintegral solution of the first row.

1	2.5	-0.5	0.5
0	2	2	2
0	9	9	9

Load the program 'insert':
load('insert')

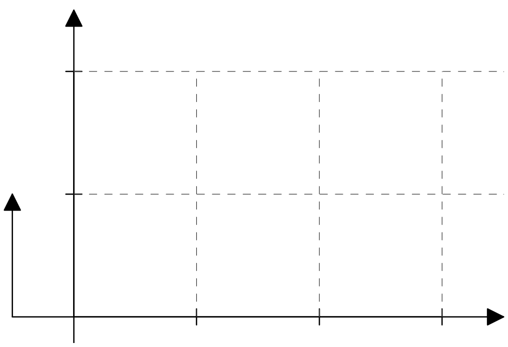
The floor of the nonintegral row ⟨2.5, -0.5, 0.5⟩ is ⟨2, -1, 0⟩.
To add row ⟨2, -1, 0⟩ with its slack column ⟨0; 0; 1; 0⟩ enter:
row = ⟨2, -1, 0⟩; >insert<
This gives

2.5	-0.5	0	0.5
2	2	0	2
2	-1	1	0
9	9	0	9

The row we inserted was ⟨2, -1, 0⟩, not ⟨2, -1, 1, 0⟩, the needed "1" for new the identity column was added automatically,

■ max $z=y$
with
a: $2x+2y\le 3$, $x,y\in N$ ... continued on other side.

■ max $z=y$
with
a: $2x+2y\leq 3$,
$x,y\in N$



Initial tableau.

x	y	r	b

Final tableau. Select y to enter and pivot.

x	y	r	b

The solution is not integral. Cut it off by adding a new floor constraint: taking the floor of the coefficients and constant.

The new constraint is: ____________________

Adding a slack gives: ____________________

New matrix.

	x	y	r	s	b
z					

New tableau. Pivot to make y and s identity columns.

	x	y	r	s	b
z					

The solution is not feasible. Restore feasibility using the dual method. In the row with the negative constant, pivot on the entry whose objective/(negative-coefficient) ratio is closest to 0.

Final tableau.

	x	y	r	s	b
z					

Answer. max $z=1$ at $y=1$, $x=0$.