Although standard problems require inequalities of the form \( ax + by \leq c \), you may use linear inequalities in any form in the problem statements below.

\[
y - 3 \leq 2(x-2) \quad \text{is a linear inequality.}
\]
\[
0 \leq x + y \leq 5
\]
can be written as 2 linear inequalities.

\[
x + y = 2r + 2s
\]
can be written as 2 linear inequalities.

\[
\text{max } z = 2x + y
\]
with
\[
x + y \leq 20
\]
\[
x \in \{4, 8, 15\}
\]
\[
y \geq 0
\]

**Variables with finite ranges.**

A decision variable \( x \in \{0, 1\} \) is a variable with a finite range. To solve decision problems, we branch two ways with constraints \( x = 0 \) and \( x = 1 \).

To solve a problem with \( x \in \{4, 8, 15\} \), branch three ways with added constraints \( x = 4, x = 8, \) and \( x = 15 \).
$	extbf{Problem: rewrite this as a problem with three decision variables } x_4, x_8, x_{15} \in \{0, 1\} \text{ instead of the variable } x.$

- $\max z = 2x + y$
  - with $x + y \leq 20$
  - $x \in \{4, 8, 15\}$
  - $y \geq 0$

- $\max z = 2(??) + y$
  - with $(??) + y \leq 20$
  - $(??) = 1 \leftarrow \text{new constraint}$
  - $x_4, x_8, x_{15} \in \{0, 1\}$
  - $y \geq 0$
max $x = 2x + y$

with

$x + y \leq 20$

$x \in \text{even}$

$y \in \text{odd}$

$x, y \geq 0$

Reduce to an integer problem.
\[ \max z = \min \{3p - q, p + q + r, -2p + 3q - r\} \]

with
\[ p, q, r \in [0,4] \]

Reduce to a linear programming problem.

\[ \max z = m \]

with
\[ m \leq ?? \]
\[ m \leq ?? \]
\[ m \leq ?? \]

\[ p, q, r \leq ?? \]
\[ p, q, r \geq ?? \]
\[ m \text{ unrestricted} \]
**The Transportation Problem**

Golf balls are made in plants $P_1, P_2, P_3, \ldots$ which produce the *supplies* and then shipped to warehouses $W_1, W_2, W_3, \ldots$ which handle the *demands*.

$s_i =$ the number of balls plant $P_i$ supplies,

$d_j =$ the number of balls warehouse $W_j$ demands,

$c_{ij} =$ cost of shipping a ball from plant $P_i$ to warehouse $W_j$,

$z =$ shipping cost.

**Assume:** $\Sigma s_i = \Sigma d_j$.

Let $x_{ij} =$ # of balls shipped from plant $P_i$ to warehouse $W_j$,

$p_i =$ the dual supply constraint variable,

$w_j =$ the dual demand constraint variable,

$o_{ij} =$ the slack in the dual constraint for $x_{ij}$.

The transportation problem has a specialized algorithm which is more efficient than the simplex method. We use a reduced 2-dimensional “tableau” with no interior coefficients.
Primal problem:

\[
\begin{align*}
\text{min } & \text{ shipping cost } \quad z = \Sigma c_{ij}x_{ij} \\
\text{with } & \\
\text{supply constraint } & p_i: \quad \Sigma_j x_{ij} = s_i, \quad i = 1, \ldots, \\
\text{demand constraint } & w_j: \quad \Sigma_i x_{ij} = d_j, \quad j = 1, \ldots, \\
x_{ij} & > 0.
\end{align*}
\]

Since each variable \( x_{ij} \) occurs in exactly one supply and one demand constraint, the dual constraint for \( x_{ij} \) is \( x_{ij}: \quad p_i + w_j \leq c_{ij} \).

Hence the dual problem is

Dual problem:

\[
\begin{align*}
\text{max } & \quad z = \Sigma s_i p_i + \Sigma d_j w_j \\
\text{with } & \\
x_{ij}: & \quad p_i + w_j \leq c_{ij}, \quad p_i, w_j \text{ unrestricted.}
\end{align*}
\]

Adding slacks gives the canonical version, \( x_{ij}: \quad p_i + w_j + o_{ij} = c_{ij}, \quad p_i, w_j \text{ unrestricted, } o_{ij} \geq 0. \)

We will use the canonical dual constraints

\[
\begin{align*}
p_i + w_j + o_{ij} &= c_{ij} \\
p_i, w_j \text{ unrestricted, } o_{ij} \geq 0.
\end{align*}
\]

as much as the original constraints

\[
\begin{align*}
\Sigma_j x_{ij} &= s_i, \\
\Sigma_i x_{ij} &= d_j.
\end{align*}
\]
We will get an initial tableau without using the two phase method.

**The initial transportation matrix**
The top border lists the warehouses \( W_1, W_2, W_3, \ldots \) which have demands. The bottom border lists their demands \( d_1, d_2, d_3, \ldots \).

The left border lists the plants \( P_1, P_2, P_3, \ldots \) which make supplies. The rightmost border lists their supplies \( s_1, s_2, s_3, \ldots \).

The number of balls \( x_{ij} \) is in a square with their shipping cost \( c_{ij} \) in the upper left corner.

Squares for parameters have uncircled zeros. Squares for basic variables have their values circled.

The initial values are given by the greedy algorithm. The text uses the min-cost or Vogel’s algorithm instead of the greedy algorithm given in the next lecture.
### Primal problem:

**min shipping cost**  \( z = \sum c_{ij}x_{ij} \)  with  

**supply constraint**  \( p_i: \ \sum_j x_{ij} = s_i, \ i = 1, \ldots, \)  

**demand constraint**  \( w_j: \ \sum_i x_{ij} = d_j, \ j = 1, \ldots, \)  \( x_{ij} \geq 0. \)

### Dual problem:

**max**  \( z = \sum s_i p_i + \sum d_j w_j \)  

**with**  \( x_{ij}: \ p_i + w_j \leq c_{ij}, \ p_i, w_j \) unrestricted.