A *Markov chain* is a graph whose edges are weighted with positive probabilities. The nodes the possible *states* of some process or machine at times, $t = 1, 2, 3, \ldots$. The probability on an edge from state $A$ to state $B$ = the probability that the next state will be $B$ if the current state is $A$.

Since there must always be a next state, the sum of the probabilities leaving a node must be 1.
A kid's net worth on any day is $0, $1, or $2. From one day to the next it may increase or decrease by one dollar. If he has $0, on the next day he will have $1 with probability 2/3, or $0 with probability 1/3. This and the other transition probabilities are pictured in the Markov chain:

Suppose we record the sequence of states for infinitely many days: day 1, 2, 3, 4, ... . In the long-run, what proportion of the time will the state be 0? If you pick a day at random, what is the probability the state will be 0? These questions are the same. Let $x$, $y$, and $z$, be the long-term probability of being in states 0, 1, and 2, respectively. These are the steady-state probabilities.
To be in a given state, you must have gotten to it from some previous state.

\[ x = \text{long-term probability of being in state} \]

\[ y = \text{long-term probability of getting to the state from the other states}. \]

These equations about the steady state probabilities plus the fact that the probabilities must total to 1 are the steady-state equations.

Write the steady-state equations for the long-term probabilities \( x, y, z \) of being in states 0, 1, 2.
Write the steady-state equations for $x, y, z$.

\begin{align*}
x &= \frac{1}{3}x + \frac{1}{2}y \\
y &= \frac{2}{3}x + \frac{1}{3}z \\
z &= \frac{1}{2}y + \frac{2}{3}z \\
x + y + z &= 1
\end{align*}

The solution is $x = \frac{3}{13}$, $y = \frac{4}{13}$, $z = \frac{6}{13}$.

From the diagram it is clear that twice as much time is spent at state 2 ($z$) as at 0 ($x$).
A company rents vans. There are three places \( A, B, C \) where vans are both rented and returned. The conditional probabilities \( T(i,j) \) that a van rented in place \( i \) is returned to place \( j \) are:

\[
\begin{array}{ccc}
A & B & C \\
\hline
A & 0.1 & 0.5 & 0.4 \\
B & 0.6 & 0.2 & 0.2 \\
C & 0.3 & 0.3 & 0.4 \\
\end{array}
\]

This matrix of transition probabilities is the \textit{transition matrix}.

\textbf{Draw the corresponding Markov chain.}

In the long-run, what are the proportions \( x, y, z \) vans at \( A, B, \) and \( C \)? The proportion of vans at \( A = \) the probability that a van will be at \( A \).

\textbf{Write the steady-state equations for} \( x, y, \) and \( z \).
A company rents vans.
There are three places $A$, $B$, $C$ where vans are both rented and returned.
The conditional probabilities $T(i,j)$ that a van rented in place $i$ is returned to place $j$ are:

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.1</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>$B$</td>
<td>0.6</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$C$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Write the steady-state equations for $x$, $y$, and $z$.

\[
x = 0.1x + 0.6y + 0.3z \\
y = 0.5x + 0.2y + 0.3z \\
z = 0.4x + 0.2y + 0.4z \\
x + y + z = 1
\]

The column totals are the same. This suggests that the probabilities of being at $A$, $B$, and $C$ should be the same; hence all should = $1/3$. Indeed $x = y = z = 1/3$ is a solution.
A shop has two drill presses, $A$ and $B$. They have two states: working, broken. If $A$ is working, the probability that it will be working the next day is .9. If it is broken, the probability that it will be fixed and working the next day is .7. For $B$ the probabilities are .95 and .6 respectively. Which works most often?

Here is the Markov chain for Machine $A$.

The transition matrices for machines $A$ and $B$ are:

$$T_A = \begin{bmatrix} x & y \\ x & .9 & .1 \\ y & .7 & .3 \end{bmatrix}, \quad T_B = \begin{bmatrix} x & y \\ x & .95 & .05 \\ y & .6 & .4 \end{bmatrix}.$$

Let $x =$ the proportion of time a machine is working,
Let $y =$ the proportion of the time it is broken. Write the steady-state equations.
\[
T_A = \begin{bmatrix} x & y \\ x & .9 & .1 \\ y & .7 & .3 \end{bmatrix}, \quad T_B = \begin{bmatrix} x & y \\ x & .95 & .05 \\ y & .6 & .4 \end{bmatrix}.
\]

The equation system for \(A\):
\[
x = .9x + .7y \\
y = .1x + .3y \\
x + y = 1
\]
Solution: \((x, y) = (7/8, 1/8)\)

\(A\) is working \(x = 7/8 = 87.5\%\) of the time.

The equation system for \(B\):
\[
x = .95x + .6y \\
y = .05x + .4y \\
x + y = 1
\]
Solution: \((x, y) = (12/13, 1/13)\)

\(B\) is working \(x = 12/13 = 92.3\%\) of the time.

Ans. \(B\)'s 92.3\% in-service time beats \(A\)'s 87.5\%. 