Math 416  Lecture 5

A professor’s computer is replaced when it dies or in its fifth year, which ever comes first. Let the state $X$ of a computer be the number of years until it is replaced. The state space is $\{1, 2, 3, 4\}$. Here are the replacement times and their probabilities for a new computer.

$p(1) = \frac{1}{10}, p(2) = \frac{2}{10}, p(3) = \frac{3}{10}, p(4) = \frac{4}{10}$

Find the transition diagram and matrix.

This problem is atypical. We usually know the current state but here the state is the number of years a computer will last. The number is well-defined number but usually unknown. More often, the current state is known but not the current outcome. For example, if a die is rolled and $X = 1$ if the number is even and 0 if not, then knowing $X$ doesn’t tell us what the outcome was.

Today’s processes will be Markov chains (time-homogeneous Markov processes).

A kid’s net worth on any day is $0$, $1$, or $2$. From one day to the next it may increase or decrease by one dollar. If it is $0$, he gets $1$ with probability 2/3. If it is $1$, he gets $1$ with probability 1/2 and loses $1$ with probability 1/2. If it is $2$, he loses $1$ with probability 1/3. In all other cases, his net worth is unchanged. Let $X_i$ be his net worth on day $i$. Here is the transition diagram and matrix:

```
0 2/3  1/2  1/3  2/3
1  1/2  0  1/2
2  0  1/3  2/3
```

The transition matrix is defined by $T(i,j) = P[X_{k+1} = j | X_k = i] = $ the probability of a transition from state $i$ to state $j$. Since Markov chains are time-homogeneous, the transition probabilities are the same for all $k$.

For this problem, the matrix is

```
0 1 2
0 1/3 2/3 0
1 1/2 0 1/2
2 0 1/3 2/3
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What is the range of possible values for entries in transition matrices?
What is the sum of a row?
What about the sum of a column?

The transition matrix gives the probabilities of transitions between states in one step. What are the probabilities after two steps? Let $T^2(i,j) = P(X_2 = j | X_0 = i) = $ the probability of being in state $j$ on day two if it starts in state $i$. $X_0$ $X_1$ $X_2$

If he starts with $0$, what is the probability of having $0$ two days later? The are several paths which start from $0$ and end at $0$ two days later. To find the total probability, add the probabilities of all paths. The probability distribution for today. Find the probability of being in state 0 tomorrow.

Theorem. For a Markov chain, the matrix $T^n(i,j)$, which gives the probability of being in state $j$ $n$ steps after being in state $i$ is $T^n = T \times T \times T \times ... \times T$.

Suppose $[p_0, p_1, p_2]$ is the row vector giving the probability distribution for today. Find the probability of being in state 0 tomorrow.

To get to state 0 tomorrow, it might be in state 0 (probability $p_0$) today and then transition to 0 (probability $T(0,0)$) tomorrow, or it might be in state 1 ($p_1$) and then transition to 0 ($T(1,0)$) or it might be in state 2 ($p_2$) and transition to 0 ($T(2,0)$).

The total is $p_0 T(0,0) + p_1 T(1,0) + p_2 T(2,0)$ = the first entry in the row vector $[p_0, p_1, p_2] \times T = [p_0, p_1, p_2] \times \begin{bmatrix} T(0,0) & T(0,1) & T(0,2) \\ T(1,0) & T(1,1) & T(1,2) \\ T(2,0) & T(2,1) & T(2,2) \end{bmatrix}$.

Theorem. If $p$ is the row vector of the current probability distribution, $pT^n$ is row vector for the distribution after $n$ steps.
Matlab notation: [1,2,3] is a row vector, [1;2;3] is a column vector.

**Definition.** For a matrix \( A \): \( X \) is an **eigenvector** and \( \lambda \) is an **eigenvalue** iff \( AX = \lambda X \).

We don’t exclude \( \lambda = 0 \). We do exclude the trivial solution \( X = 0 \) since \( A \cdot 0 = \lambda \cdot 0 \) is always true. Geometrically, \( AX = \lambda X \) says that \( AX \) and \( X \) lie on the same line. We don’t care about \( X \)'s length, just its direction. The excluded solution \( X = 0 \) has no direction. If \( X \) is an eigenvector, so is \( aX \) which lies on the same line. Thus we may simplify a fractional eigenvector \([1/2; 3/2]\) to an integral eigenvector \([1; 3]\).

1 = the identity matrix.

\[ AX = \lambda X \] iff \( AX - \lambda X = 0 \) iff \( (A - \lambda I)X = 0 \) iff \( (\lambda I - A)X = 0 \)

**Finding Eigenvectors.** Given an eigenvalue, to find the eigenvectors \( X \) for \( \lambda \), solve the matrix equation \( (A - \lambda I)X = 0 \). Apply Gaussian elimination to the augmented matrix.

**Finding Eigenvalues.** The matrix equation \( (A - \lambda I)X = 0 \) has a nontrivial solution iff \( \det(\lambda I - A) = 0 \). \( \det(\lambda I - A) \) is the **characteristic polynomial** of \( A \).

**Theorem.** The eigenvalues of a matrix \( A \) are exactly the real roots of its characteristic polynomial.

**Procedure.** To find all eigenvalues: find and factor the characteristic polynomial. List the positive eigenvalues first, then the negatives, then, possibly, 0.

**Theorem.** If the eigenvectors \( E_i \) of \( A \) form a basis for the vector space, then \( A \) is **diagonalizable** and \( A = N D N^{-1} \) where \( D \) is a diagonal matrix with eigenvalues \( \lambda_1, \lambda_2, ..., \lambda_n \) on the diagonal and \( N = [E_1 | E_2 | ... | E_n] \) where the column vector \( E_i \) is an eigenvector for \( \lambda_i \).

List repeated roots more than once. If a root has degree 3, list it 3 times.

**Definition.** For a matrix \( A \), the trace of \( A \), \( \text{trace}(A) \) is the sum of its diagonal elements.

- For \( A = \begin{bmatrix} 1 & x \\ x^2 & x^3 \end{bmatrix} \), \( \text{trace}(A) = 1 + x^3 \).

If the matrix \( A \) has parameters, e.g.,

\[ A = \begin{bmatrix} 1 - p & p \\ q & 1 - q \end{bmatrix} \]

then factoring

\[ A - \lambda I = \begin{bmatrix} 1 - p - \lambda & p \\ q & 1 - q - \lambda \end{bmatrix} \]

is problematic. An alternative way of finding the eigenvalues is to use the following theorem.

**Theorem.** If \( A = N D N^{-1} \), then \( \text{trace}(A) = \text{trace}(D) = \text{the sum of the eigenvalues of } A \). Moreover, for any \( n \), \( \text{trace}(A^n) = \text{the sum of the } n^\text{th} \text{ powers of the eigenvalues of } A \).

These equations can be used to solve for the eigenvalues. For transition matrices, we get one eigenvalue, 1, and one eigenvector, \([1; 1; ... ;1]\), for free.

**Theorem.** For any transition matrix \( T \), 1 is an eigenvalue and \([1; 1; ... ;1]\) is an eigenvector. The eigenvectors are a basis. \( T \) is diagonalizable.

**Proof.** Each row \((p_1, p_2, ..., p_n) \times (1, 1, ..., 1)^T = p_1 + p_2 + ... + p_n = 1 \). Hence \( T \times (1; 1; ...; 1) = (1; 1; ...; 1) \).

We like diagonal matrices since computing their powers is easy.

\[
D = \begin{bmatrix}
d_1 & 0 & 0 & 0 \\
0 & d_2 & 0 & 0 \\
0 & 0 & d_3 & 0 \\
0 & 0 & 0 & d_4
\end{bmatrix}
\]

then \( D^n = \begin{bmatrix}
d_1^n & 0 & 0 & 0 \\
0 & d_2^n & 0 & 0 \\
0 & 0 & d_3^n & 0 \\
0 & 0 & 0 & d_4^n
\end{bmatrix} \)

**Fact.** If \( T = N D N^{-1} \), then \( T^n = N D^n N^{-1} \)

**Proof.**

\[
T^3 = (N D N^{-1})(N D N^{-1})(N D N^{-1}) = N D (N^{-1} N) D (N^{-1} N) D N^{-1} = N D D D N^{-1} = N D^3 N^{-1}.
\]