A professor’s computer is replaced when it dies or in its fifth year, which ever comes first. Let the state $X$ of a computer be the number of years until it is replaced. The state space is $\{1, 2, 3, 4\}$. Here are the replacement times and their probabilities for a new computer.

$p(1) = \frac{1}{10}, p(2) = \frac{2}{10}, p(3) = \frac{3}{10}, p(4) = \frac{4}{10}$.

Find the transition diagram and matrix.

This problem is atypical. We usually know the current state but here the state is the number of years a computer will last. The number is well-defined number but usually unknown. More often, the current state is known but not the current outcome. For example, if a die is rolled and $X=1$ if the number is even and 0 if not, then knowing $X$ doesn’t tell us what the outcome was.

What is the range of possible values for entries in transition matrices? ($\in [?,?]$).

What is the sum of any row?

What about the sum of a column?
A kid's net worth on any day is $0, $1, or $2. From one day to the next it may increase or decrease by one dollar. Here is the transition diagram.

Find the transition diagram and matrix.
Find the matrix.
A kid's net worth on any day is $0, $1, or $2. From one day to the next it may increase or decrease by one dollar. Here is the transition diagram.

Find the transition diagram and matrix.
Find the matrix.

The transition matrix is defined by

\[ T(i,j) = P[X_{k+1} = j|X_k = i] = \text{the probability of a transition from state } i \text{ to state } j. \]

Since Markov chains are time-homogeneous, the transition probabilities are the same for all \( k \). Find the matrix.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 1/3 & 2/3 & 0 \\
1 & 1/2 & 0 & 1/2 \\
2 & 0 & 1/3 & 2/3 \\
\end{array}
\]
A transition matrix gives the probabilities of transitions between states in one step. What are the probabilities after two steps?

Let $T^2(i,j) = P(X_2 = j \mid X_0 = i)$ be the probability of being in state $j$ two days after it is in state $i$.

If he starts with $0$, what is the probability of having $0$ two days later?

$T^2(0, 0) = T(0, 0)T(0, 0) + ??$
If he starts with $0, what is the probability of having $0 two days later?

\[ T^2(0, 0) = T(0, 0)T(0, 0) + T(0, 1)T(1, 0) + T(0, 2)T(2, 0) \]

= the (0,0)-entry in the matrix product \( T \times T \).

\[
\begin{bmatrix}
T(0, 0) & T(0, 1) & T(0, 2) \\
T(1, 0) & T(1, 1) & T(1, 2) \\
T(2, 0) & T(2, 1) & T(2, 2)
\end{bmatrix}
\times
\begin{bmatrix}
T(0, 0) & T(0, 1) & T(0, 2) \\
T(1, 0) & T(1, 1) & T(1, 2) \\
T(2, 0) & T(2, 1) & T(2, 2)
\end{bmatrix}
\]

**Theorem.** For a Markov chain, the matrix \( T^n(i,j) \), which gives the probability of being in state \( j \) \( n \) steps after being in state \( i \) is \( T^n = T \times T \times T \times \ldots \times T \).
Suppose \([p_0, p_1, p_2]\) is the row vector giving the probability distribution for today.

What is the probability of being in state 0 tomorrow?

The probability of being in state 0 tomorrow = 
\[p_0 T(0, 0) + ??\]
Suppose \([p_0, p_1, p_2]\) is the row vector giving the probability distribution of the current day.

What is the probability of being in state 0 tomorrow?

There are 3 ways to get to state 0 tomorrow:

- it might be in state 0 (probability \(p_0\)) today and then transition to 0 (probability \(T(0, 0)\)),
- it might be in state 1 (probability \(p_1\)) today and then transition to 0 (probability \(T(1, 0)\)),
- it might be in state 2 (probability \(p_2\)) today and then transition to 0 (probability \(T(2, 0)\)).

The total is \(p_0 T(0, 0) + p_1 T(1, 0) + p_2 T(2, 0)\) = the first entry in the row vector \([p_0, p_1, p_2] \times T = [\ T(0, 0) & T(0, 1) & T(0, 2) \\ T(1, 0) & T(1, 1) & T(1, 2) \\ T(2, 0) & T(2, 1) & T(2, 2) \ ]\).

**Theorem.** If \(p\) is the row vector of the current probability distribution, \(p T^n\) is row vector for the distribution after \(n\) steps.
Matlab notation: $[1,2,3]$ is a row vector, $[1;2;3]$ is a column vector.

**Definition.** For a matrix $A$: $X$ is an *eigenvector* and $\lambda$ is an *eigenvalue* iff $X \neq 0$ and $AX = \lambda X$.

We don’t exclude $\lambda = 0$. We do exclude the trivial solution $X = 0$ since $A \cdot 0 = \lambda 0$ is always true.

Geometrically, $AX = \lambda X$ says that $AX$ and $X$ lie on the same line. We don’t care about $X$’s length, just its direction. The excluded solution $X = 0$ has no direction. If $X$ is an eigenvector, so is $aX$ which lies on the same line. Thus we will simplify a fractional eigenvector $[1/2; 3/2]$ to an integral eigenvector $[1; 3]$.

$I$ = the identity matrix.

$AX = \lambda X$ iff $AX - \lambda X = 0$ iff $(A - \lambda I)X = 0$ iff $(\lambda I - A)X = 0$

**Finding eigenvectors.** Given an eigenvalue, to find the eigenvectors $X$ for $\lambda$, solve the matrix equation $(A - \lambda I)X = 0$. Apply Gaussian elimination (rref) to the augmented matrix.

**Finding eigenvalues.** The matrix equation $(A - \lambda I)X = 0$ has a nontrivial solution iff $\det(\lambda I - A) = 0$.

$\det(\lambda I - A)$ is the *characteristic polynomial* of $A$. 
**Theorem.** The eigenvalues of a matrix $A$ are exactly the real roots of its characteristic polynomial.

**Procedure.** To find all eigenvalues: find and factor the characteristic polynomial. List the positive eigenvalues first, then the negatives, then, possibly, 0.

**Theorem.** If the eigenvectors $E_i$ of $A$ form a basis for the vector space, then $A$ is diagonalizable and $A = NDN^{-1}$ where $D$ is a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ on the diagonal and $N = [E_1 \mid E_2 \mid ... \mid E_n]$ where the column vector $E_i$ is an eigenvector for $\lambda_i$.

List repeated roots more than once. If a root has degree 3, list it 3 times.

**Definition.** For a matrix $A$, the trace of $A$, $\text{trace}(A)$ is the sum of its diagonal elements.
For $A = \begin{bmatrix} 1 & x \\ x^2 & x^3 \end{bmatrix}$, $\text{trace}(A) = 1 + x^3$.

If the matrix $A$ has parameters, e.g.,

$A = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$

then factoring $A - \lambda I = \begin{bmatrix} 1-p-\lambda & p \\ q & 1-q-\lambda \end{bmatrix}$ is problematic. An alternative way of finding the eigenvalues is to use the following theorem.

**Theorem.** If $A = NDN^{-1}$, then $\text{trace}(A) = \text{trace}(D) = \text{the sum of the eigenvalues of } A$. Moreover, for any $n$,

$\text{trace}(A^n) = \text{trace}(D^n) = \text{the sum of the } n^{th} \text{ powers of the eigenvalues of } A$.

These equations can be used to solve for the eigenvalues.
For transition matrices, we get one eigenvalue, 1, and one eigenvector, \([1; 1; ... ;1]\), for free.

**Theorem.** For any transition matrix \(T\), 1 is an eigenvalue, \([1; 1; ... ;1]\) is an eigenvector. The eigenvectors are a basis. \(T\) is diagonalizable.

**Proof.** Each row
\[(p_1, p_2, ..., p_n) \times (1, 1, ..., 1)^T = p_1 + p_2 + ... + p_n = 1.\]
Hence \(T \times (1; 1; ...; 1) = (1; 1; ...; 1)\).

We like diagonal matrices since computing their powers is easy.

If \(D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}\) then \(D^n = \begin{bmatrix} d_1^n & 0 & 0 & 0 \\ 0 & d_2^n & 0 & 0 \\ 0 & 0 & d_3^n & 0 \\ 0 & 0 & 0 & d_4^n \end{bmatrix}\).
**Fact.** If $T = NDN^{-1}$
then $T^n = ND^nN^{-1}$

Proof.

$T^3 = (NDN^{-1})(NDN^{-1})(NDN^{-1})$

$= ND(N^{-1}N)D(N^{-1}N)DN^{-1}$

$= NDNN^{-1}$

$= ND^3 N^{-1}$. 