Exam 1 Thursday Feb. 3, covers Lectures 1-6, not Lecture 7.

For a transition matrix $T$, the $n$-step transition probabilities are given by $T^n$. The limit the system approaches as time goes to $\infty$ is $T^\infty = \lim_{n \to \infty} T^n$. Diagonalization makes this limit easy to compute.

$$\lim_{n \to \infty} T^n = \lim_{n \to \infty} N D^n N^{-1} = N (\lim_{n \to \infty} D^n) N^{-1}.$$

If $\lambda = 1$, $\lim_{n \to \infty} \lambda^n = 1$.
If $\lambda \in (-1,1)$, $\lim_{n \to \infty} \lambda^n = 0$.
Otherwise, the limit is undefined.

- $\lim_{n \to \infty} \frac{1}{2}^n = 0$, $\lim_{n \to \infty} 2^n = \infty$, $\lim_{n \to \infty} (-1)^n = \text{undefined}$.

When defined, $\lim_{n \to \infty} D^n$ is all 0’s except for some 1’s on the diagonal.
Find the characteristic polynomial, the eigenvalues and eigenvectors for \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). Find \( D, N \) and \( A^4, \lim_{n \to \infty} A^n \).
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**Characteristic polynomial:** 
\[
\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).
\]

**Eigenvalues:** \( \lambda = 1 \) deg. 1, \( \lambda = -1 \) deg. 1.

- Let \( X = [x; y] \) be the eigenvector for \( \lambda = 1 \). \( AX = \lambda X \) iff 
  
  \[(\lambda I - A)X = 0 \] \iff \[((1)I - A)X = 0].\]

  The augmented matrix is 
  \[
  \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
  \]

  Solution: \( x = y \), \( y \) arbitrary. \([x, y]^T = [y, y]^T = y[1, 1]^T \). Picking \( y = 1 \) gives the eigenvector \([1, 1]^T\).

- Let \( v = [x; y] \) be the eigenvector for \( \lambda = -1 \). \( Av = \lambda v \) iff 
  
  \[(\lambda I - A)v = 0 \] \iff \[((-1)I - A)v = 0].\]

  The augmented matrix is 
  \[
  \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
  \]

  Solution: \( x = y \), \( y \) is arbitrary. \([x, y]^T = [-y, y]^T = y[-1, 1]^T \).
Picking $y=1$ gives the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

$$N^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$ In SciLab, $N^{-1} = \text{inv}(N)$

$$A = NDN^{-1}, \quad A^4 = ND^4N^{-1} = N \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^4 N^{-1} = N \begin{bmatrix} 1^4 & 0 \\ 0 & (-1)^4 \end{bmatrix} N^{-1}$$

$$= N \cdot I \cdot N^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\lim_{n \to \infty} A^n = \lim_{n \to \infty} D^n = \begin{bmatrix} \lim_{n \to \infty} (1^n) & 0 \\ 0 & \lim_{n \to \infty} (-1)^n \end{bmatrix} = \text{undefined.}$$
**DEFINITION.** Given a Markov chain with times $k$ in the time set $\{0, 1, 2, 3, 4, \ldots\}$ and a state $j$ in the state space $S$, $PT(j)$, the **passage time** to $j$ = the first time $k \in \{0, 1, 2, 3, 4, \ldots\}$ such that $X_k = j$. (This is a multi-state version of the geometric time-to-success which involved only two states).

Let $F_k(i, j) = P[PT(j) = k \mid X_0 = i]$

= the probability that the first arrival to $j$ occurs $k$ steps after the initial state $i$.

Starting from the initial state $i$, $F_k(i, j) = the$ probability of arriving at state $j$ for the first time after exactly $k$ steps.

What is the relation between $T^k(i, j)$ and $F_k(i, j)$?

$F_k(i, j) \leq T^k(i, j)$? $F_k(i, j) = T^k(i, j)$? $F_k(i, j) \geq T^k(i, j)$?
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The transition probability $T^k(i, j)$ additionally includes $k$-step paths which visit $j$ more than once, hence $F_k(i, j) \leq T^k(i, j)$.

If $j$ is accessible from every state, $j$ will eventually be reached with probability 1 and $F_k(i, j)$ is a probability distribution on the times $k$. 
Case for initial state $i = \text{destination } j$, then $PT(j) = 0$ and
$F_0(i,j) = F_0(j,j) = 1$, $F_{n+1}(i,j) = F_{n+1}(j,j) = 0$.
Case for $i \neq j$.
Subcase $k=1$. $F_1(i,j) =$ the probability of going from state
$i$ to state $j$ in exactly 1 step $= T(i,j)$.
Subcase $k>1$. $PT(j) = k$ iff $k$ is the first time $X_k = j$ iff
$X_1 \neq j, X_2 \neq j, X_2 \neq j, ..., X_{k-1} \neq j$ and, $X_k = j$.
$F_k(i,j) = P[X_1 \neq j, X_2 \neq j, ..., X_{k-1} \neq j, X_k = j | X_0 = i]$  
= the sum of the probabilities of going from $i$ to a state
$x \neq j$ in 1 step and then from $x$ to $j$ in exactly $k-1$ steps
$= \sum_{x \in S \setminus \{j\}} T(i,x)F_{k-1}(x,j)$.

**Theorem.** $F_0(i,j) = 1$ iff $i = j$. Otherwise,
$F_1(i,j) = T(i,j)$,
$F_k(i,j) = \sum_{x \in S \setminus \{j\}} T(i,x)F_{k-1}(x,j)$.

The process by which the formula was deduced is more important than the formula.
To get into calculus, one must first pass an assessment exam or, if one fails the assessment exam, take precalculus until one passes it. (a) What is the probability of needing $n$ semesters to get into calculus? (b) What is the expected number of semesters?

Let the state space be \( \{A, P, C\} \) (for assessment, precalculus and calculus). Suppose he takes \( n+1 \) steps to get from \( A \) to \( C \). Since the first step (assessment exam) doesn’t take a semester, the number of semesters = \( n \). Here’s the diagram with.

\[
\begin{array}{c}
\text{A} \\
\text{P} \\
\text{C}
\end{array}
\]

\[ \begin{array}{ccc}
1/4 & 3/4 & 2/3 \\
1/3 & 1/3 & 1/3 \\
2/3 & 2/3 & 2/3
\end{array} \]

\[ F_1(A, C) = \frac{1}{4}, \quad F_1(A, P) = \frac{3}{4}, \quad F_1(P, C) = \frac{2}{3}, \]
\[ \frac{T(P, P) = \frac{1}{3}, \quad T^2(P, P) = \left(\frac{1}{3}\right)^2, \quad \ldots, \quad T^n(P, P) = \left(\frac{1}{3}\right)^n, \]
\[ F_{n+1}(A, C) = F_1(A, P)T^{n-1}(P, P)F_1(P, C) = \frac{3}{4}\left(\frac{1}{3}\right)^{n-1}\frac{2}{3}. \]

Answer: The probability of getting in to calculus after exactly \( n \) semesters is \( F_{n+1}(A, C) \).
If \( n = 0 \), \( F_{n+1}(A, C) = \frac{1}{4} \), (prob. of passing assessment exam).
If \( n > 0 \), \( F_{n+1}(A, C) = \frac{3}{4} \left( \frac{1}{3} \right)^{n-1} \frac{2}{3} = \frac{1}{2 \cdot 3^{n-1}} \), (probability of failing assessment exam and then passing precalculus after \( n-1 \) failures).

Let \( f(n) \) be the expected number of steps to get into calculus from state \( n \in \{A, P, C\} \).

\[
\begin{align*}
  f(C) & = 0, \\
  f(P) & = 1 \frac{2}{3} + (1 + f(P)) \frac{1}{3}, \\
  \therefore f(P) & = \frac{2}{3} + \frac{1}{3} f(P), \quad f(P) - \frac{1}{3} f(P) = 1, \quad f(P) = \frac{3}{2}, \\
  f(A) & = 1 \frac{1}{4} + (1 + f(P)) \frac{3}{4} = \frac{1}{4} + (1 + \frac{3}{2}) \frac{3}{4} = \frac{17}{8}.
\end{align*}
\]

Since the assessment doesn’t take a semester, the # of semesters = # of steps – 1. The expected number of semesters to get into calculus is \( n-1=17/8-1=9/8 \).
Suppose one starts at \( n = 20 \) and your goal is to reach \( n = 30 \). At each step you jump forward a random (with bound 30), equally likely amount. If you are at 26, then after jumping you will be at 27, 28, 29, or 30 all with equal probability 1/4 = 1/(distance between 26 and 30).

Let the state of a number be its distance \( i \) to 30.

Number: 20 21 22 23 24 25 26 27 28 29 30
State: 10 9 8 7 6 5 4 3 2 1 0.
The state space is \( \{0, 1, ..., 10\} \).

If you are a distance \( i \) from 30, there are \( i \) numbers to which you may jump, all have probability 1/\( i \).

If you start at \( n = 20 \) (state \( i = 10 \)), what is the expected number of jumps to reach \( n = 30 \) (state \( i = 0 \))?
Let \( f(i) \) be the expected number of jumps to state 0 \((n = 30)\) from state \( i \) \((n = 30 - i)\).

If \( i = 0 \), no jumps are needed and \( f(0) = 0 \).
If \( i = 1 \), there is exactly one possible jump is \( f(1) = 1 \).
If \( i > 1 \) you either

get to state \( i = 0 \) \((n = 30)\) in one jump \(1/i\)
or first jump \(1/i\) to an intermediate point
with state \( j \in \{1, 2, 3, ..., i-1\} \). And then jump \(1/i\) to the final state.

\[
f(i) = 1 + \sum_{j=1}^{i-1} \frac{1}{i} + f(j) = \frac{1}{i} + \frac{1}{i} \sum_{j=1}^{i-1} 1 + \frac{1}{i} \sum_{j=1}^{i-1} f(j)
= \frac{1}{i} + \frac{1}{i} (i - 1) + \frac{1}{i} \sum_{j=1}^{i-1} f(j)
= \frac{1}{i} + 1 - \frac{1}{i} + \frac{1}{i} \sum_{j=1}^{i-1} f(j)
\]

\[
f(i) = 1 + \sum_{j=1}^{i-1} f(j)
\]

Calculate some values and look for a pattern.

\[
f(0), f(1), f(2), f(3)
\]

\[
0, 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, ...
\]

Answer: \( f(i) = \sum_{k=1}^{i} \frac{1}{k} \).