For a random Markov chain, in the limit as time goes to infinity, a state is either visited infinitely often, or is visited only finitely often, in which case, after some final visit, it is not visited again.

Suppose state \( x \) is visited infinitely often. Suppose there are arrows leading from \( x \) to \( a, b, \) and \( c \) with probabilities 1/2, 1/3, and 1/6. Then, by the Strong Law, with probability 1, the proportion of times it goes from \( x \) to \( a, b, \) and \( c \) respectively will be 1/2, 1/3, and 1/6, in the limit. (Technical check, the sequence of states in a Markov chain are not independent but the sequence of states following visits to \( x \) are.)

- Starting from \( A \), where will a random chain of events eventually go?

![Diagram](https://via.placeholder.com/150)

Starting at \( A \), by the Strong Law, it will eventually leave \( A \) (with probability 1). Once it leaves \( A \), it never returns. Likewise for \( F \). If it goes to \( E \), it gets stuck there forever. Otherwise it cycles through the set \( \{B, C, D, E\} \) each of which will, with probability 1 be visited infinitely often.

For a probability \( p \), \( \lim_{n \to \infty} p^n = 0 \) if \( p < 1 \); =1 if \( p = 1 \).

**Definition.** A state is absorbing if you can’t check out (the \( E \) above is absorbing). If it is visited only in finite often (with probability 1), it is transient (states \( A, F \) are transient). If, once visited, it is visited infinitely often (with probability 1), it is recurrent (\( B, C, D \) are recurrent).

**Convention:** Omit arrows with probability 0.

**Definition.** For states \( i \) and \( j, i \to j, j \) is reachable from \( i \), iff there is a path from \( i \) to \( j \). A set of states is closed iff no arrow leads from a state in the set to a state outside the set iff one can not leave the set. A set of states is irreducible iff it is closed and no proper subset is closed.

In any diagram, \( \emptyset \) and the entire state space \( S \) are trivially closed. For random events, with probability 1, one eventually ends up in one of the irreducible sets.

- In the diagram above, \( \{E\} \) and \( \{B, C, D\} \) are the irreducible sets, \( \{F, B, C, D\} \) is closed but not irreducible.

A closed set is a subspace of the state space. Its transition matrix is a submatrix of the full matrix. Its matrix \( T \) is regular iff for some \( n, T^n \) has only nontrivial entries (neither 0 or 1).

**Theorem.** Let \( i, j, k \) be states of a Markov process. Then
- (a) \( i \to j \) and \( j \to k \) implies \( i \to k \).
- (b) If \( j \) is recurrent and \( j \to k \), then \( k \to j \) and \( k \) is recurrent.
- (c) If \( j \to k \) always implies \( k \to j \), then \( j \) is recurrent.
- (d) If, for some \( k, j \to k \) but not \( k \to j \), then \( j \) is transient.
- (e) The set of states reachable from \( j \) is closed.

If \( j \) is recurrent, the set of states reachable from \( j \) is the recurrence class of \( j \). It is irreducible.

- In the example, the recurrence class of \( E \) is \( \{E\} \); the recurrence class of \( C \) is \( \{B, C, D\} \).

States are in the same recurrence class are mutually reachable and any infinite chain in the class will include all members of the class infinitely often (with probability 1).

**Theorem.** Starting from a state \( j \), the expected number of return visits to \( j \) equals \( \sum_{n=1}^{\infty} T^n(j,j) \).

**Proof.** Let \( I_n = 1 \) if the state at time \( n \) is \( j \), 0 if not. The expected number of return visits, starting from \( j \), is

\[
E[\sum_{n=1}^{\infty} I_n|X_0 = j] = \sum_{n=1}^{\infty} E[I_n|X_0 = j] = \sum_{n=1}^{\infty} (1(T^n(j,j)) + 0(1 - T^n(j,j))) = \sum_{n=1}^{\infty} T^n(j,j).
\]

**Theorem.** Suppose \( i \) and \( j \) are states. Then
- (a) If \( j \) is not reachable from \( i \), then for all \( n, T^n(i,j) = 0 \).
- (b) If \( j \) is transient, then \( \lim_{n \to \infty} T^n(i,j) = 0 \).

Now suppose \( i \) and \( j \) belong to the same recurrence class and the matrix for the class is regular.

Let \( \pi_j = \lim_{n \to \infty} T^n(j,j) \). This is the “long-term” probability of being at \( j \). Let \( M_j = \) the average time between visits of \( j \). For a random chain \( \omega \) of states, let \( N_n(j,\omega) = \) total number of visits to \( j \) up to time and including \( n \). Thus \( N_n(j,\omega)/n = \) the percentage of time during \([0,n]\) spent in state \( j \).

Clearly, \( \pi_j \) small
- the proportion of time at \( j \) is small
- on average, the number of visits \( N_n(j,\omega) \) is small
- the average time \( M_j \) between visits is long.

**Theorem.** If \( i \) and \( j \) are in the same recurrence class and the class matrix is regular, then

\[
\pi_j = \lim_{n \to \infty} T^n(i,j) = T^\infty(i,j) \text{ (} \pi_j \text{ doesn’t depend on } \omega \text{)}
\]

\[
\pi_j = \lim_{n \to \infty} N_n(j,\omega)/n \text{ (with probability 1 for random } \omega \text{),} \pi_j = 1/M_j .
\]
Now suppose $C_1, C_2, \ldots, C_m$ are the recurrence classes and $S$ is the set of transient states. Order the states so that those in $C_1$ come first, those in $C_2$ second, \ldots, and those in $S$ are last. Then the transition matrix $T$ will have submatrices along the diagonal, one for each recurrence class, a row for each transient state and zeros elsewhere.

Here $C_1 = \{E\}$ and $C_2 = \{B, C, D\}$ are the recurrence classes. $S = \{A\}$ is the transient class. Note the block structure of the matrix $T$. Which areas are necessarily 0?

\[
\begin{array}{c|cccc}
\text{C} & E & B & C & D \\
\hline
E & 1 & 0 & 0 & 0 \\
B & 0 & 0 & 0 & 1 \\
C & 0 & 1/2 & 1/2 & 0 \\
D & 0 & 0 & 1 & 0 \\
A & 1/4 & 1/4 & 1/4 & 0 & 1/4 \\
\end{array}
\]

$T^n$ and $T^\infty$ will have the same block structure as $T$. The transient columns of $T^\infty$ will be 0 by last time’s theorem: $j$ transient $\Rightarrow \lim_{n\to\infty} T^n(i, j) = 0$. Here it is the last column.

$T^\infty =
\[
\begin{array}{c|cccc}
\text{C} & E & B & C & D \\
\hline
E & ? & 0 & 0 & 0 \\
B & 0 & ? & ? & ? & 0 \\
C & 0 & ? & ? & ? & 0 \\
D & 0 & ? & ? & ? & 0 \\
\end{array}
\]

To find $T^\infty$, calculate the limits of the submatrices along the diagonal and then the limits of the transient rows.