Theorem. If $T$ is a regular matrix for an irreducible space or submatrix for a recurrence class, then:

- The rows of $T^\infty$ are identical. Let $\pi$ be the common row. Its $j^{th}$ entry = the proportion $\pi_j$ of time in state $j$.
- The entries of a row $\pi$ are probabilities that total to 1.
- For each row $\pi$, $\pi \times T = \pi$.

Proof. By the previous theorem, for all $i$,

$T^\infty(i,j) = \lim_{n \to \infty} T^n(i,j) = \pi_j$. Since $\pi_j$ doesn’t depend on $i$, all rows are the same.

For each row of $T^n$ the elements are probabilities which total to 1. This fact remains true in the limit.

Finally $T^\infty = \lim_{n \to \infty} T^n = \lim_{n \to \infty} T^{n+1} = (\lim_{n \to \infty} T^n)T = T^\infty T$. Hence $T^\infty = T^\infty T$. Hence for each row $\pi$ of $T^\infty$,

$\pi = \pi T$.

Use $\pi = \pi T$ and $\sum \pi_j = 1$ to calculate $\pi$.

For this diagram, calculate $T^n$ for the class $\{B, C, D\}$.

$T = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Let $\pi = (x, y, z) = $ a row of $T^\infty$.

Solve $(x, y, z) = (x, y, z) T$ and $x + y + z = 1$:

$x = \frac{1}{2} y$, $y = \frac{1}{4} y + z$, $z = x$, $x + y + z = 1$.

Ignore the last equation. Solving the first three gives $y = 2x$, $y = 2z$, $x = z$. $x = 1, y = 2, z = 1$. Divide by the total $1 + 2 + 1 = 4$ to make the last equation true. Thus $x = 1/4, y = 1/2, z = 1/4$. Multiplying the solution by a constant won’t violate the first three equations since they are homogeneous.

Hence $T^\infty = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

The $j^{th}$ entry = the proportion $\pi_j$ of time spent in the $j^{th}$ state. Hence, inside the recurrence class $\{B, C, D\}$, the chain spends $1/4$ of its time at B, $1/2$ of its time at C, and $1/4$ of its time at D. This is reasonable: when at B and D you must leave, half of the time when at C, you stay.

RECALL: $\pi_j = T^\infty(j, j) = T^\infty(i, j)$ = the long-term probability of being in state $j$. $\pi = (\pi_1, \pi_2, ..., \pi_m)$ = the long-term probability distribution. To calculate $\pi$, solve the equations $\pi = \pi T$ and $\sum \pi_j = 1$. Applying $T$ to both sides gives $\pi T = \pi T^2$. Putting the equations together gives $\pi = \pi T^2$. Iterating gives $\pi = \pi T^n$ for any $n$. Since this long-term probability distribution $\pi$ remains unchanged, it is called the steady-state (or equilibrium) distribution and its equations $\pi = \pi T$ and $\sum \pi_j = 1$ are the steady-state equations.

### Inventory Theory

A car dealer makes $1000 net on the sale of a jeep (his selling price – his buying price). He pays $100/month in interest for every jeep on the lot. For each month, the number of customers who want jeeps has the following probability distribution.

<table>
<thead>
<tr>
<th>number $D$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>1/8</td>
<td>1/8</td>
<td>4/8</td>
<td>2/8</td>
</tr>
</tbody>
</table>

For the $n^{th}$ month, let $D_n$ = the demand = the number of jeeps customers want to buy, let $X_n$ = the number of jeeps on the lot at the beginning of the month.

$. \quad $ his monthly interest payment is $100 \times X_n$.

The number of jeeps he sells during the month is $\leq$ the number $X_n$ available to sell and $\leq$ the number $D_n$ customers want. Hence the number he sells = $\min(X_n, D_n)$ and his sales for the month = $1000 \times \min(X_n, D_n)$.

If the number of jeeps on the lot at the end of the month is $s \leq 1$, he increases the total number on the lot to $S = 4$. Otherwise the number is left unchanged.

Why doesn’t he always increase the number to 4? Why wait until inventory is 1 or 0 to increase the number? The ordering process often involves shipments and paperwork. Hence businesses prefer to consolidate orders and reduce the number of times orders have to be made. Rather than replenish inventory continuously, orders are issued only when inventory dips below a set limit “s” (1 for us).

When that limit is reached, the inventory is brought up to some level “S” (4 for us) which is often to amount of space available for that product.

How many jeeps does he have on the lot at the beginning of a month? How much interest does he pay for jeeps per month? What are his monthly sales of jeeps?

The intended meaning of “How many jeeps does he have ...” is the long-term or steady-state probability distribution $\pi$ of $X_n$. The rule “If the number of jeeps ... Otherwise ... unchanged.” becomes

$X_{n+1} = 4$ if $X_n - D_n \leq 1$. $X_{n+1} = X_n - D_n$ if not.

The state space for $X_n = \{2, 3, 4\}$. Why can’t there be 0, 1?

Let $(x, y, z) = (\pi_2, \pi_3, \pi_4) = \pi$ where $\pi_i$ is the long-term probability of there being $i$ jeeps on the lot at the beginning of the month.

Likewise, “How much interest does ...” and “What are his month sales ...” mean the expected long-term averages.

The transition matrix for $X_n$ is
The steady state equations are \((x, y, z) = (x, y, z)T\).

\[
\begin{align*}
8x &= 1x + 1y + 4z \\
8y &= 0x + 1y + 1z \\
8z &= 7x + 6y + 3z
\end{align*}
\]

The augmented matrix for this system is

\[
\begin{bmatrix}
1 & 0 & 7 \\
1 & 1 & 6 \\
4 & 1 & 3
\end{bmatrix}
\]

The steady state equations are \((x, y, z) = (x, y, z)T\).

\[
\begin{align*}
x &= \frac{1}{8} \\
y &= \frac{8}{1} \\
z &= \frac{7}{8}
\end{align*}
\]

Thus \(x = \frac{29}{49}z\), \(y = \frac{1}{7}z\), \(z\) arbitrary.

To make things integral, pick \(z\) to be 49.

Thus \(x = 29\), \(y = 7\), \(z = 49\).

To make \(x + y + z = 1\) true, divide \(x, y, z\) by the total \(29 + 7 + 49 = 85\).

Thus \(x = \frac{29}{85}, y = \frac{7}{85}, z = \frac{49}{85}\).

Hence the steady-state probability distribution is

\[
\begin{bmatrix}
2 & 3 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
29/85 & 7/85 & 49/85
\end{bmatrix}
\]

This is the best answer to the question “How many jeeps does he have at the beginning of a month”.

The monthly interest (long-term average) is

\[
\begin{align*}
\text{Monthly profit} &= \text{net sales} - \text{monthly interest} = 1789.71 - 323.53 = 1466.18
\end{align*}
\]

The checksum = 1466.18

\[
\begin{align*}
\text{Long-term average net sales} &= \$1000 \sum_{i,j} \min(j, i)j \cdot P[D = j]P[X = i]
\end{align*}
\]

To be systematic, draw a \(D-X\) matrix with a row for each demand value \(D\) and a column for each state value \(X\). Then each entry will be \(\min(j, i)j \cdot P[D = j]P[X = i]\). To simplify calculations, factor out the \(D\) and \(X\) probability denominators. In the matrix, we omit the row for \(D = 0\) since it will be a row of zeros. In the matrix entries we’ve omitted the denominators since they are all the same. The first entry \(1(1)(29)\) is actually \(\min(1, 2)2 \cdot P[D = 1]P[X = 2] = 1(\frac{1}{8})(\frac{29}{85})\), the last term is \(3(\frac{3}{8})(\frac{49}{85})\).

Thus \(x = \frac{29}{85}, y = \frac{7}{85}, z = \frac{49}{85}\).