**Theorem.** If $N_t$ is the number of events up to time $t$ and if the $n$th event arrives at time $T_n$:

For a given $t$, \[ P[N_t = n] = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \]
\[ E[N_t] = \lambda t. \]

For a given $n$, \[ P[N_t < n] = P[T_n > t] = \sum_{k=0}^{n-1} e^{-\lambda t} (\lambda t)^k/k!, \]
\[ E[T_n] = n/\lambda. \] □
When flipping a coin, the previous history of outcomes has no effect on the next outcome — the coin has no memory. Likewise for Poisson and exponential distributions. (See Problem 1, Homework 4.)

**Theorem.**

For a Poisson distribution $N_t$, the number of events between times $t$ and $t + s = N_{t+s} - N_t$. This number is independent of $t$ and of the previous history $N_u : u \leq t$.

\[
E(N_{t+s} - N_t) = E(N_s) = E(N_t)
\]

For an exponential distribution, $T_{k+1} - T_k$, the expected time between $T_k$ and the next event, is independent of $T_k$ and all earlier events $T_1, T_2, T_3, ..., T_{k-1}$.

\[
E(T_{n+k} - T_k) = E(T_n) = E(T_k)
\]
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This number is independent of \( t \) and of the previous history \( N_u : u \leq t \).

\[
E(N_{t+s} - N_t) = E(N_s - N_0) = E(N_s - 0) = E(N_s).
\]

For an exponential distribution, \( T_{k+1} - T_k \), the expected time between \( T_k \) and the next event, is independent of \( T_k \) and all earlier events \( T_1, T_2, T_3, ..., T_{k-1} \).

\[
E(T_{n+k} - T_k) = E(T_n - T_0) = E(T_n). \quad \square
\]
Email arrives according to a Poisson process with a rate \( \lambda = 10 \) a day. If there are 5 emails in the first half of a day, what is the probability there will be 5 in the last?

Rate per half a day = ??

Probability there will be 5 in the last half day?

Continuing with \( \lambda = 10 \) a day. If 10 emails arrive in the first half of the day, what is the expected number for the whole day?
Email arrives according to a Poisson process with a rate $\lambda = 10$ a day. If there are 5 emails in the first half of a day, what is the probability there will be 5 in the last?

Poisson processes are memoryless, what happened the first half of a day doesn’t affect the second half. The probability depends only on the length of the interval, not when it starts. Hence

$$P[5 \text{ in second half } | \ 5 \text{ in first half}] = P[5 \text{ in half a day}] = P[N_{1/2} = 5] = \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

$$= e^{-10(\frac{1}{2})} (10(\frac{1}{2}))^5 / 5! = e^{-5} \frac{5^5}{5!}$$

Continuing with $\lambda = 10$ a day. If 10 emails arrive in the first half of the day, what is the expected number for the whole day?

The expected number for the last half depends only on the rate and duration of the time interval. It is $E(N_{1/2}) = \lambda t = (10)(1/2) = 5$. Hence the expected number for the whole day is $10 + 5 = 15$.

Given that 10 emails arrived during the day, what is the probability that 3 arrived in the first half?
Given that 10 emails arrived during the day, what is the probability that 3 arrived in the first half?

\[ P[3 \text{ first half} \mid 10 \text{ total}] \]

\[ = P[3 \text{ first half}, 10 \text{ total}] / P[10 \text{ total}] \]

\[ = P[3 \text{ first half}, 7 \text{ second half}] / P[10 \text{ total}] \]

\[ = P[3 \text{ first half}] P[7 \text{ second half}] / P[10 \text{ total}] \]

\[ = P[N_{1/2} = 3] P[N_{1/2} = 7] / P[N_1 = 10] \]

\[ = [(\lambda \frac{1}{2})^3 e^{-(\lambda \frac{1}{2})}/3!] [(\lambda \frac{1}{2})^7 e^{-(\lambda \frac{1}{2})}/7!] / [(\lambda 1)^{10} e^{-(\lambda 1)}/10!] \]

\[ = (\frac{1}{2})^{10} 10!/(3!7!) = (\frac{1}{2})^3 (\frac{1}{2})^7 (\frac{10}{3}) = P_B(x = 3 \mid p = \frac{1}{2}, n = 10). \]

Note: the \( \lambda \)'s all cancelled off. When the total number of events is known, the rate \( \lambda \) at which they occur is irrelevant. The probability \( P_B(x = 3 \mid p = \frac{1}{2}, n = 10) \) is

the probability of \( x = 3 \) successes (success = arriving in the first half)

in a binomial process of \( n = 10 \) trials

whose probability of success is \( p = \frac{1}{2} \).
The number of arrivals is Poisson iff the time of the next arrival is exponentially distributed with the same rate $\lambda$. But if we know that there will be $n$ arrivals during the time period $[0, t]$, then the $n$ arrival times are uniformly distributed over $[0, t]$ and the exponential arrival rate $\lambda$ is not relevant.

Except for the fact that they occur in increasing order, the $n$ arrival times $T_1, T_2, T_3, ..., T_n$ have the same distribution as that of $n$ points chosen completely at random from $[0, t]$. Arrival times can generated with the correct probability distribution by randomly choosing $n$ points in $[0, t]$ and then reindexing the $n$ numbers to be in increasing order.

**Theorem.** Given that there are $n$ arrivals during the time period $[0, t]$, the uniform (it depends on $t$ but not $t_1, ..., t_n$) joint density function for $T_1 = t_1, T_2 = t_2, ..., T_n = t_n$ is $f(t_1, t_2, ..., t_n) = h$ for $0 < t_1 < t_2 < ... < t_n < t$, where $h = ??$ ...

... $h$ is directly?/inversely? proportional to $n$?

... $h$ is directly?/inversely? proportional to $t$?

**Proof.**
**Theorem.** Given that there are \( n \) arrivals during the time period \([0, t]\), the uniform (it depends on \( t \) but not \( t_1, \ldots, t_n \)) joint density function for \( T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n \) is \( f(t_1, t_2, \ldots, t_n) = n!/t^n \) for \( 0 < t_1 < t_2 < \ldots < t_n < t \).

**Proof.** Let \( h \) be the value of this uniform function. Picking \( n \) points from \([0, t]\) is equivalent to picking one point from \([0, t]^n\) which has volume \( t^n \). However only \( 1/n! \) of the points in \([0, t]^n\) are in increasing order. Hence the volume of the region under \( h \) is \( ht^n/n! \). Set this equal to 1 and solve for \( h \):

\[
f(t_1, t_2, \ldots, t_n) = h = n!/t^n \cdot f(t_1, t_2, \ldots, t_n) = h = n!/t^n.
\]

**Remark:** The observation that the proportion of area occupied by \( \{(t_1, \ldots, t_n) : t_1 < t_2 < \ldots < t_n \} \) is \( 1/n! \) that of \([0, t]^n\) can help simplify integrals with uniform density functions \( c \).

\[
\int_{0<x<y<z<5} c = \int_0^5 \int_0^z \int_0^y c \, dx \, dy \, dz = ??
\]
**Remark:** The observation that the proportion of area occupied by \( \{(t_1, ..., t_n): t_1 < t_2 < ... < t_n\} \) is \( 1/n! \) that of \([0, t]^n\) can help simplify integrals with uniform density functions \( c \).

\[
\int_{0 < x < y < z < 5} c = \int_0^5 \int_0^z \int_0^y c \, dx \, dy \, dz = \int_0^5 \int_0^5 \int_0^5 c \, dx \, dy \, dz / 3! = c5^3 / 3!.
\]
Emails arrive as before with a Poisson distribution at a rate of \( \lambda = 10 \) a day. I want to check my email only twice a day. Each message must be answered on the same day it arrives. I want to minimize the delay between a message’s arrival and the time it is answered. **When should I check my email?**

In order to answer every email, the second check must be at the end of the day. Intuition says that the first check should be in the middle of the day. We verify this.
Emails arrive as with a Poisson distribution at a rate of $\lambda = 10$ a day. I want to check my email only twice a day. Each message must be answered on the same day it arrives. I want to minimize the delay between a message’s arrival and the time it is answered? When should I check my email?

Let $[0, 1]$ represent one day’s length. The time of the second email check is 1. Let $t$ be the time of the first email check. If the $i^{th}$ email arrives at time $T_i$, then either

- it arrives in $[0, t]$ and is answered at the first email check with a delay of $t - T_i$ or
- it arrives in $(t, 1]$ and is answered at the second email check with a delay of $1 - T_i$. 
Calculate the expected answering delay for emails arriving in the \([0, t]\) time interval.

\(N_t\) is the number of emails in this period and the expected delay is

\[
E(\sum_{i=1}^{N_t} (t - T_i)) = \sum_{n=0}^{\infty} E(\sum_{i=1}^{n} (t - T_i) \mid N_t = n) P[N_t = n]
\]

Given that \(N_t = n \frac{t}{2}\),

by the theorem, \(x = T_i\) has a uniform distribution on \([0, t]\).

\[\therefore\] for one email, the density function is \(f(x) = \frac{1}{t}\).

\[\therefore\] the expected value of each delay is

\[
E(t - T_i \mid N_t = n) = \int_{0}^{t} (t - x)(1/t)dx = \int_{0}^{t} [1 - x/t]dx
\]

\[= [x - x^2/2t]_{0}^{t} = t - t/2 = t/2.\]

\(\therefore\) the expected delay for all \(n\) emails is \(n \frac{t}{2}\).

This is just for the case \(N_t = n\), totaling all cases gives

\[
\sum_{n=0}^{\infty} E(\sum_{i=1}^{n} (t - T_i) \mid N_t = n) P[N_t = n] =
\]

\[
\sum_{n=0}^{\infty} (n \frac{t}{2}) P[N_t = n] = \frac{t}{2} \sum_{n=0}^{\infty} n P[N_t = n] = \frac{t}{2} E(N_t)
\]

\[= \frac{t}{2} (\lambda t) = \frac{t}{2} (10t) = \frac{10t^2}{2} = 5t^2.\]

\(E(N_t) = \lambda t\) by a Lect. 14 theorem.
Likewise the expected delay for emails in the \([t, 1]\) time interval is \(5(1 - t)^2\) since the interval length is \(1 - t\) rather than \(t\).

The total delay is
\[
5t^2 + 5(1 - t)^2 = 5\left(t^2 + (1 - 2t + t^2)\right) = 5(2t^2 - 2t + 1).
\]
To minimize the delay, set the derivative to 0.

\[
0 = 5(4t - 2), \quad 4t = 2, \quad t = 1/2.
\]

Hence the first email check should be in the middle of the day as was expected.
Birth and death processes, queueing theory

In arrival processes, the state only jumps up, in a *birth-death* process, it can either jump up or down by one unit. A birth-death process counts the number of objects in a *queue* to which items can be added or deleted.

In arrival processes, the arrival rate $\lambda$ is constant and does not depend on the state $j$ (the size of the population).

This isn’t the case with birth/death processes. When the population is 0, the death rate $\mu_0=0$ (there is no one left to die). If there is an upper bound $M$ on the population size, the state space of population sizes is $\{0, 1, 2, ..., M\}$. When the population size is maximum, no more births are possible, hence the birth rate $\lambda_M=0$. □