Math 416  Lecture 11
Birth and death processes, queueing theory
In arrival processes, the state only jumps up, in a
birth-death process, it can either jump up or down by one
unit. A birth-death process counts the number of objects
j=N, in a queue to which items can be added or deleted.
In an arrival process, the arrival rate \( \lambda \) is constant and
does not depend on the state \( j \) (the size of the population). In a
birth/death process, both births and deaths may depend on
the state \( j \). When the population is 0, the death rate \( \mu_0=0 \)
(the there is no one left to die). If there is an upper bound \( M \) on
the population size, the state space of population sizes is
\( \{0, 1, 2, ..., M\} \). When the population size is maximum, the
birth rate \( \lambda_M=0 \) since no more births are possible.

Examples of queues.
- The customer queue in front of a checkout counter. A
  birth occurs when a customer joins the line. A death
  occurs when the customer has been served or when a
  customer leaves the line for another checkout lane.
- The resident population of Hawaii is a queue. A birth
  occurs when a new resident arrives or is born. A death
  occurs when a resident dies or leaves Hawaii.
- The world-wide population of monk seals. Births and
deaths are the only way the population changes. In this
case, when the population \( j=0 \) not only is the death
rate \( \mu_j=0 \), but also the birth rate \( \lambda_j=0 \) (extinction has
occurred).
- The number of occupied tables in a restaurant. This
  queue has an upper bound, the number \( M \) of tables.
  When all tables are occupied tables, \( j=M \), and \( \lambda_j=0 \)
since no more births (occupied tables) are possible.

Two births might occur at the same time but since time \( t \)
is continuous, this has probability 0 and we ignore such
cases when computing probabilities.

In Poisson processes the time between arrivals is
exponentially distributed with some arrival rate \( \lambda \). The
obvious analog for birth-death processes would be to
measure the time between changes — either births or
deaths. However, the birth rates and death rates may
differ. In a steady-state system, the birth and death rates
are the same. If the birth rate exceeds the death rate, there
will be a population increase. The birth \( \lambda_j \) and death rates
\( \mu_j \) may depend on the size \( j \) of the population. The
death rate is 0 when the population is 0 and the birth rate is
often proportional to the population size \( j \).

Definition. A birth-death process is a continuous-time
counting process \( N_t \) with birth rates \( \lambda_j \) and a death rates \( \mu_j \)
which depend on the current state \( j=N_t \). \( t \in [0, \infty) \) = timelset
\( N_t \in \{0, 1, 2, 3, \ldots \} \) or
\( \{0, 1, 2, 3, \ldots, M\} \) if there is an upper bound \( M \). Unless
specified otherwise, assume \( N_0=0 \).

Let \( T_n \) be the sequence of state-change times. \( T_0=0, T_1=\)
the time of the first birth/death, \( T_2=\) the time of the
second state change, ... As in any Poisson process, the
times \( T_{n+1} - T_n \) between state changes are exponentially
distributed with a combined rate of \( \lambda_j + \mu_j \) where \( j \) is the
state \( j=N_t \) at time \( t = T_n \).

Let \( X_t \) be the sequence of states \( X_n = X_{T_n} \) at the times \( T_n \)
of a state change (either a birth or death). \( X_0=0, X_1=\)
the state immediately after the first birth/death, \( X_2=\)
state at the time of the second state change, ... . Since
\( N_{T_n} = X_n, N_1 = X_{T_1} \) when \( t \in [T_n, T_{n+1}] \).

Given a time \( t \) and the current state \( j=N_w \), the time \( T_{n+1} \)
and the type (birth or death) of the next state change
depends only on the current state \( j \). It is independent of
the time \( t \) and independent of the previous history of
times \( (T_1, T_2, ..., T_{n-1}) \) and states \( (X_1, X_2, ..., X_{n-1}) \).
The combined rate of transitions (births plus deaths) is \( \lambda_j +
\mu_j \). At transitions, the probabilities of births and deaths =
\( P[X_{n+1} = j + 1 | X_n = j] = \lambda_j / (\lambda_j + \mu_j) \) and
\( P[X_{n+1} = j - 1 | X_n = j] = \mu_j / (\lambda_j + \mu_j) \) respectively.

Let \( B_t \) be the sequence of birth times. \( B_0=0, B_1=\) the time
of the first birth, ... . The times between births are
exponentially distributed with a birth rate of \( \lambda_j \).

Let \( D_t \) be the sequence of death times. The times between
deaths are exponential with a death rate of \( \mu_j \).

A hospital has \( M \) beds. Whne beds are available, patients
are admitted at a rate \( \lambda \). Each individual patient dies or is
released at a rate of \( \mu \). If there are \( j \) beds occupied, what
are \( \lambda_j \) and \( \mu_j \)?

If \( j < M \), then \( \lambda_j = \lambda \).
If \( j = M \), \( \lambda_j = 0 \) (no more beds are available).
The more occupied beds there are, the higher the rate of
deaths/releases. Hence we expect \( \mu_j \) to be proportional to \( j \)
rather than constant. If there are \( j \) beds occupied and each
occupant is released/dies at a rate of \( \mu \), then the total
death/release rate is \( \mu_j = j \mu \).

Definition. Let \( p_{ij}(t) = P[N_t = j \mid N_0 = i] \).

Let \( \lambda_i = \lim_{t \to \infty} p_{ij}(t) \).
As was the case for \( \pi_i \) and regular Markov chains, under
general regularity conditions, the limit \( \lambda_i = \lim_{t \to \infty} p_{ij}(t) \)
exists and is independent of the starting state \( j \). \( \lambda_j \) is the
eventual proportion of time spent in state \( j \).

In a care-home, 1/8 of the patients alive at time \( t \) will
eventually leave the home (as opposed to dying in the
home). The number of patient arrivals \( N_t \) is Poisson with
rate \( \lambda \). Find the number \( D_{2t} \) of patients at time \( 2t \) who
will eventually leave. The obvious answer is \( \frac{1}{8} \lambda(2t) \).

Interpret “1/8 of the patients leave” as an expected value:

\[
E[D_1 | N_t = n] = \frac{1}{8} n.
\]

\[
E[D_{2t}] = \sum_{n=0}^{\infty} E[D_{2t} | N_{2t} = n] P[N_{2t} = n]
= \sum_{n=0}^{\infty} \frac{1}{8} n P[N_{2t} = n]
= \frac{1}{8} \sum_{n=0}^{\infty} n P[N_{2t} = n] = \frac{1}{8} E[N_{2t}] = \frac{1}{8} \lambda(2t).
\]

**Kolmogorov Equations**

From Lecture 16: **Definition.** Let \( p_{ij}(t) = P[N_t = j | N_0 = i] \).

Let \( p_j = \lim_{t \to \infty} p_{ij}(t) \).

As was the case for \( \pi_j \), under rather general regularity conditions, the limit exists and is independent of the initial state \( i \). \( p_j = \lim_{t \to \infty} p_{ij}(t) \) is eventual proportion of time spent in state \( j \). For this lecture and the homework, assume the needed regularity conditions hold.

We estimate change using a small step of size \( h \) where \( h \) is sufficiently small that the probability of two or more birth/death events during time \( h \) is in small enough that such double events may be ignored in estimating population changes during time intervals of size \( h \). For differentiable functions and small \( h \):

\[
f(t + h) - f(t) \approx f'(t)h.
\]

The amount of change is proportional to the rate of change and to the length of the time period.

For a birth/death process the rates of change \( f'(x) \) are the birthrates \( \lambda_j \) and death rates \( \mu_j \) where \( j \) is the population size. Thus over a time period of length \( h \) the amount of change due to births is \( \lambda_j h \) and the change due to deaths is \( \mu_j h \).

\[
p_j(t) = \text{the probability of being in state } j \text{ at time } t.
\]

\[
= \text{the probability the population has size } j \text{ at time } t.
\]

\[
p_j(t + h) - p_j(t)
= \text{the change in the probability of being in state } j \text{ during the time interval } [t, t + h].
\]

There are two ways a population can cease to be of size \( j \): via a birth or death. There are two ways a population can get to be of size \( j \): a birth to a population of size \( j-1 \) or a death to a population of size \( j+1 \). Thus

\[
\lambda_{j-1}, \lambda_j \leftarrow \mu_j \leftarrow \mu_{j+1}\]

\[
p_j(t + h) - p_j(t) \approx p'_j(t) h
= -p_j(t) \lambda_j h - p_j(t) \mu_j h + p_{j-1}(t) \lambda_{j-1} h + p_{j+1}(t) \mu_{j+1} h
= h(-p_j(t) \lambda_j - p_j(t) \mu_j + p_{j-1}(t) \lambda_{j-1} + p_{j+1}(t) \mu_{j+1})
\]

\[
p_j(t + h) - p_j(t) = \frac{h}{\lambda_j} p_j(t) \lambda_j h - \frac{h}{\lambda_{j-1}} p_{j-1}(t) \lambda_{j-1} h + \frac{h}{\mu_j} p_j(t) \mu_j h + \frac{h}{\mu_{j+1}} p_{j+1}(t) \mu_{j+1} h
= \frac{h}{\lambda_j} p_j(t) \lambda_j h + \frac{h}{\lambda_{j-1}} p_{j-1}(t) \lambda_{j-1} h + \frac{h}{\mu_j} p_j(t) \mu_j h + \frac{h}{\mu_{j+1}} p_{j+1}(t) \mu_{j+1} h
\]

Taking the limit as \( h \) goes to 0 gives

\[
p'_j(t) = -p_j(t) \lambda_j - p_j(t) \mu_j + p_{j-1}(t) \lambda_{j-1} + p_{j+1}(t) \mu_{j+1}
\]

When \( j = 0 \), \( \mu_j = 0 \), there is no smaller population. Thus the equation becomes

\[
p'_0(t) = -p_0(t) \lambda_0 + p_1(t) \mu_1
\]

Under rather general regularity conditions, in the limit, as \( t \to \infty \), the population probability distribution approaches a stable distribution \( p_j \). In a stable distribution the population remains unchanged. Hence \( p_j(t) = 0 \). Thus \( p_j(t) \) is a constant \( p_j \).

Hence the long-term the probabilities \( p_0, p_1, p_2, ... \) of the population having sizes \( j = 0, 1, 2, ... \) satisfy the following **Kolmogorov** steady state equations:

\[
0 = -p_0 \lambda_0 + p_1 \mu_1,
\]

\[
...,
0 = -p_j \lambda_j - p_j \mu_j + p_{j-1} \lambda_{j-1} + p_{j+1} \mu_{j+1},
\]

\[
... .
\]

Thus \( p_1 \) is proportional to \( p_0 \).

\( p_2 \) is proportional to \( p_1, p_0 \) and hence to just \( p_0, p_{j+1} \) is proportional to \( p_0 \).

Hence all are proportional to \( p_0 \) and hence all are proportional to each other.

In particular \( p_j \) is proportional to \( p_{j-1} \).

Is \( p_j \) directly or inversely proportional to \( \lambda_{j-1} \)?

Is \( p_j \) directly or inversely proportional to \( \mu_j \)?

Write \( p_j \) in terms of \( p_{j-1} \), \( \lambda_{j-1} \) and \( \mu_j \).