Birth and death processes, queueing theory

In arrival processes, the state only jumps up.

In a \textit{birth-death} process, it can either jump up or down by one unit.

A birth-death process counts the number of objects \( j = N_t \) in a \textit{queue} to which items can be added or deleted. The state is the size \( j \) of the population.

In arrival processes, the arrival rate \( \lambda \) is constant and does not depend on the state \( j \). In birth-death processes, both birth rates \( \lambda_j \) and death rates \( \mu_j \) may depend on the state \( j \).

When the population is 0, the death rate \( \mu_0 = ?? \)
If there is an upper bound $M$ on the population size, the state space of population sizes is $\{0,1,2,\ldots,M\}$. When the population size is maximum, the birth rate $\lambda_M = ??$
Examples of queues.

- The customer queue in front of a checkout counter. A birth occurs when a customer joins the line. A death occurs when the customer has been served or when a customer leaves the line for another checkout lane.

- The resident population of Hawaii is a queue. A birth occurs when a new resident arrives or is born. A death occurs when a resident dies or leaves Hawaii.

- The world-wide population of monk seals. Births and deaths are the only way the population changes. In this case, when the population \( j = 0 \) not only is the death rate \( \mu_j = 0 \), but also the birth rate \( \lambda_j = 0 \) (extinction has occurred).

- The number of occupied tables in a restaurant. This queue has an upper bound, the number \( M \) of tables. When all tables are occupied tables, \( j = M \), and \( \lambda_j = 0 \) since no more births (occupied tables) are possible.

What happens when two births occur at the same time? How does this affect the calculation of probabilities?
In Poisson processes the time between arrivals is exponentially distributed with some arrival rate $\lambda$. The obvious analog for birth-death processes would be to measure the time between changes — either births or deaths. However, the birth rates and death rates may differ.

In a steady-state system, how are birth and death rates related?

When the population is 0, the death rate must be 0 but the birth rate could be nonzero.

If the birth rate exceeds the death rate, what happens?

The birth $\lambda_j$ and death rates $\mu_j$ may depend on the size $j$ of the population.

The birth rate is often proportional to the population size $j$. 
**Definition.** A *birth-death* process is a continuous-time counting process $N_t$ with birth rates $\lambda_j$ and a death rates $\mu_j$ which depend on the current state $j = N_t$.

$t \in [0, \infty) =$ timeset

$N_t \in$ the state space of possible counts $= \{0, 1, 2, 3, \ldots \}$ or $\{0, 1, 2, 3, \ldots, M\}$ if there is an upper bound $M$. Unless specified otherwise, assume $N_0 = 0$.

Let $T_n$ be the sequence of state-change times.

$T_0 = 0$,

$T_1 =$ the time of the first birth/death,

$T_2 =$ the time of the second state change, ... . **What is the rate of these state changes?**

Let $X_n$ be the sequence of states $X_n = N_{T_n}$ at the state-change times $T_n$ (a birth or a death).

$X_0 = 0$,

$X_1 =$ the state (population size) immediately after the time of the first birth/death,

$X_2 =$ the population size at the time of the second state change, ... .
Since $N_{T_n} = X_n$, $N_t = X_n$ when $t \in [T_n, T_{n+1})$.

The times $T_{n+1} - T_n$ between state changes are exponentially distributed with a combined rate of $\lambda_j + \mu_j$ where $j = X_n$ is the state at time $T_n$.

Given a time $t$ and the current state $j = N_t$, the time $T_{n+1}$ and the type (birth or death) of the next state change depends only on the current state $j$. It is independent of the time $t$ and independent of the previous history of times $(T_1, T_2, ..., T_{n-1})$ and states $(X_1, X_2, ..., X_{n-1})$.

Suppose again that the birth rate is $\lambda_j$ and the death rate is $\mu_j$ where the state $j = N_t$.

The combined rate of transitions (births plus deaths) is $\lambda_j + \mu_j$.

For each transition with state $X_n$ and time $T_n$,

What is the probability that the next transition is a birth?
$P[X_{n+1} = j + 1 \mid X_n = j] = ??$

What proportion of the state changes are deaths?
$P[X_{n+1} = j - 1 \mid X_n = j] = ??$
Let $B_n$ be the sequence of birth times. $B_0 = 0$, $B_1 =$ the time of the first birth, ... . The times between births are exponentially distributed with a birth rate of $\lambda_j$.

Let $D_n$ be the sequence of death times. The times between deaths are exponential with a death rate of $\mu_j$.

A hospital has $M$ beds. When beds are available, patients are admitted at a rate $\lambda$. Each individual patient dies or is released at a rate of $\mu$. If there are $j$ beds occupied, what are $\lambda_j$ and $\mu_j$?

If $j = M$, $\lambda_j =$ ??
If $j < M$, then $\lambda_j =$ ??
If $j = 0$, $\mu_j =$ ??
If $j > 0$, then $\mu_j =$ ??

The more occupied beds there are, the higher the rate of deaths/releases. Hence we expect $\mu_j$ to be proportional to $j$ rather than constant.

If there are $j$ beds occupied and each occupant is released/dies at a rate of $\mu$, then the total death/release rate is $\mu_j = j\mu$. 
**DEFINITION.** Let $p_{ij}(t) = P[N_t = j \mid N_0 = i]$.

Let $p_j = \lim_{t \to \infty} p_{jj}(t)$.

As was the case for $\pi_j$ and regular Markov chains, under general regularity conditions, the limit $p_j = \lim_{t \to \infty} p_{ij}(t)$ exists and is independent of the starting state $j$.

$p_j$ is the eventual proportion of time spent in state $j$.

In a care-home, $1/8$ of the patients die in a period of $t$ years. The number patient arrivals $N_t$ is Poisson with rate $\lambda$. Find the number $D_{2t}$ of patients who will die in a period of $2t$ years.

The obvious answer is $\frac{1}{8} \lambda(2t)$. What if we interpret “$1/8$ of the patients die in a period of $t$ years” as an expected value:

$$E[D_t \mid N_t = n] = \frac{1}{8} n?$$

$$E[D_{2t}] = \sum_{n=0}^{\infty} E[D_{2t} \mid N_{2t} = n] P[N_{2t} = n]$$

$$= \sum_{n=0}^{\infty} \frac{1}{8} n P[N_{2t} = n] =$$

$$= \frac{1}{8} \sum_{n=0}^{\infty} n P[N_{2t} = n]$$

$$= \frac{1}{8} E[N_{2t}] = \frac{1}{8} \lambda(2t)$$
Kolmogorov Equations

From Lecture 16:

**Definition.** Let $p_{ij}(t) = P[N_t = j | N_0 = i]$. Let $p_j = \lim_{t \to \infty} p_{jj}(t)$.

As was the case for $\pi_j$, under rather general regularity conditions, the limit exists and is independent of the initial state $i$. $p_j = \lim_{t \to \infty} p_{ij}(t) = \text{eventual proportion of time spent in state } j$. For this lecture and the homework, assume the needed regularity conditions hold.

We calculate these “long-term” probabilities the same way we did $\pi_j$. But now the time variable is continuous. Instead of a system of equations which calculate possible next steps, we have a differential equations which calculate the rates of change.

We estimate change using a small step of size $h$ where $h$ is sufficiently small that the probability of two or more birth/death events during time $h$ is small enough that such double events may be ignored in estimating population changes during time intervals of size $h$.

For differentiable functions and small $h$: 
\[ f(t+h) - f(t) \approx f'(t)h \]
The amount of change is proportional to the rate of change and to the length of the time period.

For a birth/death process the rates of change \( f'(x) \) are
- the birthrates \( \lambda_j \) and
- the death rates \( \mu_j \) where \( j \) is the population size.

For a time period of length \( h \), write the following in terms of \( \lambda_j, \mu_j \) and \( h \).

The amount of change due to births. ??
The change due to deaths. ??

\[ p_j(t) = \text{the probability of being in state } j \text{ at time } t. \]
\[ = \text{the probability the population has size } j \text{ at time } t. \]

\[ p_j(t+h) - p_j(t) \]
\[ = \text{the change in the probability of being in state } j \text{ during the time interval } [t, t+h]. \]

What are the two ways a population can cease to be of size \( j \)?

What are the two ways a population can get to be of size \( j \)?
For a birth/death process the rates of change $f'(x)$ are the birthrates $\lambda_j$ and death rates $\mu_j$ where $j$ is the population size. Thus over a time period of length $h$ the amount of change due to births is $\lambda_j h$ and the change due to deaths is $\mu_j h$.

$p_j(t) = \text{the probability of being in state } j \text{ at time } t.$

$\text{= the probability the population has size } j \text{ at time } t.$

$p_j(t + h) - p_j(t)$

$\text{= the change in the probability of being in state } j \text{ during the time interval } [t, t + h].$

The two ways a population can cease to be of size $j$: via a birth or death.

The two ways a population can get to be of size $j$:

via a birth to a population of size $j-1$

or via a death in a population of size $j+1$.

Thus

\[
\begin{array}{ccc}
(j-1) & \xrightarrow{\lambda_{j-1}} & j \\
\mu_j & \xleftarrow{} & \mu_{j+1} \\
& \xrightarrow{\lambda_j} & (j+1)
\end{array}
\]
\[ p_j(t + h) - p_j(t) \approx p_j'(t)h = ?? \]

\[ \frac{p_j(t+h) - p_j(t)}{h} = ?? \]

Taking the limit as \( h \) goes to 0 gives

\[ p_j'(t) = ?? \]

When \( j = 0 \), \( \mu_j = 0 \), there is no smaller population. Thus the equation becomes

\[ p_0'(t) = \]

Under rather general regularity conditions, in the limit, as \( t \to \infty \), the population probability distribution approaches a stable distribution \( p_j \). In a stable distribution the population remains unchanged. Hence \( p_j'(t) = 0 \). Thus \( p_j(t) \) is a constant \( p_j \).
Taking the limit as $h$ goes to 0 gives

$$p_j(t) = -p_j(t)\lambda_j - p_j(t)\mu_j + p_{j-1}(t)\lambda_{j-1} + p_{j+1}(t)\mu_{j+1}$$

When $j=0$, $\mu_j=0$ and there is no smaller population. Thus the equation becomes

$$p'_0(t) = -p_0(t)\lambda_0 + p_1(t)\mu_1$$

Under rather general regularity conditions, in the limit, as $t \to \infty$, the population probability distribution approaches a stable distribution $p_j$. In a stable distribution the population remains unchanged. Hence $p'_j(t) = 0$. Thus $p_j(t)$ is a constant $p_j$. 
Hence $p'_j(t) = 0$.

And $p_j(t)$ is a constant $p_j$.

Thus the equations

$$p'_j(t) = -p_j(t)\lambda_j - p_j(t)\mu_j + p_{j-1}(t)\lambda_{j-1} + p_{j+1}(t)\mu_{j+1}$$

$$p'_0(t) = -p_0(t)\lambda_0 + p_1(t)\mu_1$$

become ???
Hence $p'_j(t) = 0$. Thus $p_j(t)$ is a constant $p_j$. Thus the equations

$$p'_j(t) = -p_j(t)\lambda_j - p_j(t)\mu_j + p_{j-1}(t)\lambda_{j-1} + p_{j+1}(t)\mu_{j+1}$$

$$p'_0(t) = -p_0(t)\lambda_0 + p_1(t)\mu_1$$

become

$$0 = -p_0\lambda_0 + p_1\mu_1, \ldots,$$

$$0 = -p_j\lambda_j - p_j\mu_j + p_{j-1}\lambda_{j-1} + p_{j+1}\mu_{j+1}, \ldots$$

These equations for the long-term the probabilities $p_0, p_1, p_2, \ldots$ of the population having sizes $j=0, 1, 2, \ldots$ are the Kolmogorov steady state equations:

Thus $p_1$ is proportional to $p_0$.

$p_2$ is proportional to $p_1, p_0$ and hence to just $p_0, p_{j+1}$ is proportional to $p_0$.

Hence all are proportional to $p_0$ and hence all are proportional to each other.

In particular $p_j$ is proportional to $p_{j-1}$.

Is $p_j$ directly or inversely proportional to $\lambda_{j-1}$?

Is $p_j$ directly or inversely proportional to $\mu_j$?

Write $p_j$ in terms of $p_{j-1}, \lambda_{j-1}$ and $\mu_j$. 
Kolmogorov steady state equations

(a) \(0 = -p_0 \lambda_0 + p_1 \mu_1\)

(b) \(0 = -p_j \lambda_j - p_j \mu_j + p_{j-1} \lambda_{j-1} + p_{j+1} \mu_{j+1}\)

Solve (a) for \(p_1\)

Solve (b) for \(p_2\).

Write \(p_2\) in terms of \(p_0\)

Write \(p_2\) in terms of \(p_1\)

Write \(p_3\) in terms of \(p_0\)

Write \(p_3\) in terms of \(p_2\)

Since all probabilities must total to 1,

\(p_0 + p_1 + p_2 + ... = 1\)

Solve for \(p_0\).