Math 416 Lecture 13 not on Exam 2

RECALL. Suppose there are infinitely many lines, the arrival times are Poisson with rate $\lambda$ and the service times have cumulative probability $G(x)$. Suppose $X_t$ is the number of customers in the system at time $t$. Then the expected number of customers in the system at time $t$ was shown to be

$$E[X_t] = \lambda t p = \lambda t \int_0^t (1 - G(x)) dx = \lambda \int_0^t (1 - G(x)) dx$$

Now suppose that the service times are exponentially distributed with rate $\mu$. Let $S_i$ be the service time for the $i$th arrival. Then

$$1 - G(x) = P[S_i > x] = \int_x^\infty e^{-\mu t} dt$$

$$= -\int_x^\infty e^{u} du = -[e^u]_x^\infty = -[0 - e^{-\mu x}] = e^{-\mu x}.$$ 

Thus

$$E[X_t] = \lambda \int_0^t e^{-\mu x} dx =$$

$$= \int_0^t e^{-\mu x} dx = \mu (1 - e^{-\mu t}).$$

As time goes to $\infty$, the expected number being served goes to $\lambda \mu$. This is reasonable, the number in the system should be proportional to the number of arrivals and inversely proportional to the rate at which they are served.

**Theorem.** Let $S$ be a continuous random variable with state space $[0, t]$ or $[0, \infty)$ and with cumulative distribution function $g(x)$ and cumulative distribution function $G(x)$. Note that $G(t) = 1$ when the state space is $[0, t]$. Then

$$E[S] = \int_0^t (1 - G(x)) dx$$

**Proof.** First, the case for $[0, t]$: 

$$[x(1 - G(x))]' = (1 - G(x)) - xG'(x) = (1 - G(x)) - xg(x).$$

$$\therefore \int_0^t [x(1 - G(x))] dx = \int_0^t (1 - G(x)) dx$$

$$\therefore \int_0^t 1 - G(x)] dx = E[S]$$

$$\therefore E[S] = \int_0^t (1 - G(x)) dx.$$

Replacing $t$ by $\infty$ gives the case for $[0, \infty)$:

$$E[S] = \int_0^\infty (1 - G(x)) dx.$$

**Corollary.** Since $G(x) = P[X \leq x]$, $1 - G(x) = P[S > x]$ the cases become $E[S] = \int_0^\infty P[S > x] dx$ or $\int_0^\infty P[S > x] dx$.

**Theorem.** Let $S$ be a discrete random variable with state space $\{0, 1, 2, 3, \ldots \}$. Then $P[S = n] = \Sigma_{n=0}^\infty P[S > n]$. 

**Proof.** Picture $np(n)$ as $n p(n)$'s stacked on top of each other.

**Poisson/Poisson/s Queues**

Cars arrive at a toll plaza with $s$ toll booths. The arrival times and service times are exponential (hence the counts are Poisson) with arrival rate $\lambda$ and service rate $\mu$. Assume arrivals and services are independent. Since the time between arrivals is exponential, no matter what time $t$ we pick, the time to the next arrival has the same expected value (the process doesn’t remember how much time has already been spent waiting).

This is a birth/death process. At any time $t$, the population size $j = X_t$ is the total number of cars in the plaza (either waiting or being served) at time $t$.

On an arrival (birth), $X_t$ increases by 1; on the completion of a service (death) $X_t$ decreases by 1. For a population of size $j$, the birth rate is the arrival rate: $\lambda = \lambda_j$.

The death rates are harder.

If $X_t = 0$ deaths are not possible and the death rate is $\mu = 0$.

**Case 1** $1 \leq j \leq s$. Thus all $j$ cars on the plaza are currently being served. Thus the next change will either be an arrival (birth rate $= \lambda$) or a service completion for one of the $j$ cars being served (death rate $= \mu$). The overall rate of changes (arrivals + service) is $\lambda + j \mu$. The probability the next change is an arrival (birth) is $\lambda / (\lambda + j \mu)$; the probability the next change is a death (service completion) is $j \mu / (\lambda + j \mu)$.

**Case 2** $s < j$. In this case only $s$ of the $j$ are being serviced and the rest wait in the queue. Thus the next change will either be an arrival (rate $= \lambda$) or service completion for one of the $s$ cars being served (rate $= \mu$). The overall rate of change is $\lambda + s \mu$. The probability the next change is birth is $\lambda / (\lambda + s \mu)$; the probability the next change is a death is $s \mu / (\lambda + s \mu)$.

Combining the two cases gives: The birth rates are $\lambda_j = \lambda$. The death rates are $\mu_j = j \mu$ if $0 \leq j \leq s$.

$\mu_j = s \mu$ if $j \geq s$.

Here are Kolmogorov’s equations (Lecture 12) for the long-term probability $p_j$ that the number of cars on the plaza is $j$:

$$p_0 = 1/(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} + \ldots)$$

The denominator is

$$1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} + \ldots + \frac{\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_{s-1}}{\mu_1 \mu_2 \ldots \mu_s} + \ldots$$

$$= 1 + \frac{\lambda}{\mu} + \frac{\lambda \lambda}{\mu \mu_2} + \frac{\lambda \lambda \lambda}{\mu \mu_2 \mu_3} + \ldots + \frac{\lambda \lambda \ldots \lambda}{\mu \mu_2 \ldots \mu_s} + \frac{\lambda \lambda \ldots \lambda}{\mu \mu_2 \ldots \mu_s} + \ldots + \frac{\lambda \lambda \ldots \lambda}{\mu \mu_2 \ldots \mu_s} + \ldots$$

$$= \Sigma_{j=0}^\infty \frac{\lambda^j}{\mu^j} + \Sigma_{k=1}^\infty \frac{\lambda^j}{s \mu^{j+k}}$$

$$= 1/[\Sigma_{j=0}^\infty (\frac{\lambda}{\mu})^j]\frac{\lambda}{\mu} + \frac{\lambda}{s \mu} \Sigma_{k=1}^\infty (\frac{\lambda}{s \mu})^k].$$

The last sum is $\frac{\lambda}{s \mu} [1 - \frac{\lambda}{s \mu}] = \frac{\lambda}{s \mu - \lambda}$. Hence

$$p_0 = 1/[\Sigma_{j=0}^\infty (\frac{\lambda}{\mu})^j]\frac{\lambda}{\mu} + \frac{\lambda}{s \mu - \lambda}].$$

$$P_j = \frac{\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_j}{\mu_1 \mu_2 \ldots \mu_j} P_0$$

$$= (\frac{\lambda}{\mu})^j \frac{1}{j!} P_0$$ if $j \leq s$. 
\[ p_j = \left( \frac{j}{\mu} \right)^j \frac{1}{j!} s^{j-s} P_0 \text{ if } s < j. \]

**Theorem.** To keep the queue from becoming infinitely large we must have \( \lambda < \mu s. \)

**Proof.** To keep the queue length bounded, large size queues must have small probabilities. As the queue size \( j \) goes to \( \infty \), the probability \( p_j \) must go to 0.

\[
\lim_{j \to \infty} p_j = \lim_{j \to \infty} \left( \frac{j}{\mu} \right)^j \frac{1}{j!} s^{j-s} P_0
= \lim_{j \to \infty} \left( \frac{j}{\mu s} \right)^j \frac{j!}{s!} P_0
= \frac{s^j}{s!} P_0 \lim_{j \to \infty} \left( \frac{j}{\mu s} \right)^j.
\]

This is 0 iff \( \frac{j}{\mu s} < 1 \) iff \( \lambda < \mu s. \)

This just says the obvious: the birthrate (number of arrivals) must be less than the death rate (number which can be served).