Recall from last time. Poisson Arrival Queues

Suppose the queue has type Poisson/G(x)/∞.
Thus the arrivals are Poisson with some rate $\lambda$.
There are infinitely many lines a customer can choose from (practically speaking, this just means that there are almost always more lines than customers not that there are $\infty$ many lines).
The service times have some cumulative probability function $G(x)$.

$\therefore G(x) = P[S_i \leq x].$
$\therefore 1 - G(x) = P[S_i > x].$

Since arrivals are Poisson, $P[N_t = n] = e^{-\lambda t} (\lambda t)^n / n!$
Since there are always more lines than customers the customers never wait in line.
$\therefore$ the time the customer is in the system
  = the time it takes for the customer to be served.
Customer $i$ leaves at time $T_i + S_i$.

Let $p$ be the probability that the $i^{th}$ arrival is still in the system at time $t$. Then

$p = P[T_i + S_i > t] = \ldots = (1/t) \int_0^t (1 - G(x)) dx.$
Given that there are \( n \) arrivals at time \( t \),
the probability that exactly \( k \) of the \( n \) arrivals are still being
served at time \( t \)

= the binomial probability of \( k \) successes out of \( n \) trials

where the probability \( p \) of success is the probability an
arrival is still being served.

\[ \therefore \text{ the probability that exactly } k \text{ of the } n \text{ arrivals are still }
\text{ being served at time } t \]

\[ P[X_t = k \mid N_t = n] = \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k} \]

where \( p = \frac{1}{t} \int_0^t (1 - G(x)) \, dx \).

To get the unconditional probability that there are exactly \( k \) customers still in the system at time \( t \), split on \( n \).

\[ P[X_t = k] = \ldots = \frac{e^{-p\lambda t}(p\lambda t)^k}{k!} = \text{ the Poisson distribution with rate } p\lambda t. \]

Since \( P[X_t = k] \) = the Poisson distribution with rate \( p\lambda t \),
the expected number of customers in the system at time \( t \) is

\[ E[X_t] = \lambda tp = \lambda t \left( \frac{1}{t} \right) \int_0^t (1 - G(x)) \, dx = \lambda \int_0^t (1 - G(x)) \, dx \]
Now suppose that the service times are exponentially distributed with rate $\mu$.

Then

$$1 - G(x) = P[S_i > x] = \int_x^\infty \mu e^{-\mu t} \, dt \quad \text{let} \quad u = -\mu t, \quad du = -\mu \, dt, \quad dt = \frac{-du}{\mu}$$

$$= \mu \int_{-\mu x}^\infty e^u \left(\frac{-du}{\mu}\right) = -\int_x^\infty e^u \, du = -[e^u]_{-\mu x}^\infty = -[0 - e^{-\mu x}] = e^{-\mu x}.$$ 

$$\therefore E[X_t] = \lambda \int_0^t e^{-\mu x} \, dx \quad \text{let} \quad u = -\mu x, \quad dx = \frac{-du}{\mu}$$

$$= \lambda \int_0^{-\mu t} e^u \left(\frac{-du}{\mu}\right) = -\frac{\lambda}{\mu} \int_0^{-\mu t} e^u \, du = -\frac{\lambda}{\mu} [e^{-\mu t} - 1] = \frac{\lambda}{\mu} (1 - e^{-\mu t}).$$

As time $t$ goes to $\infty$, $e^{-\mu t}$ goes to 0, and the expected number being served goes to $\lambda/\mu$.

This is reasonable, the number in the system should be proportional to the number of arrivals and inversely proportional to the rate at which they are served.
Theorem. Let $S$ be a continuous random variable with state space $[0, t]$ or $[0, \infty)$ and with continuous density function $g(x)$ and cumulative distribution function $G(x)$. Note that $G(t) = 1$ when the state space is $[0, t]$. Then

$$E[S] = \int_0^t (1 - G(x)) \, dx \text{ or } \int_0^\infty (1 - G(x)) \, dx$$

Proof. First, the case for $[0, t]$.

$$[x(1 - G(x))]' = (1 - G(x)) - xG'(x) = (1 - G(x)) - xg(x).$$

$$\therefore \int_0^t [x(1 - G(x))]' \, dx = \int_0^t [1 - G(x)] \, dx - \int_0^t xg(x) \, dx$$

$$\therefore [x(1 - G(x))]_0^t = \int_0^t [1 - G(x)] \, dx - E[S]$$

$$\therefore 0 = \int_0^t [1 - G(x)] \, dx - E[S]$$

$$\therefore E[S] = \int_0^t [1 - G(x)] \, dx$$

Replacing $t$ by $\infty$ gives the case for $[0, \infty)$:

$$E[S] = \int_0^\infty [1 - G(x)] \, dx.$$

Corollary. Since $G(x) = P[X \leq x]$, $1 - G(x) = P[S > x]$ the cases become

$$E[S] = \int_0^t P[S > x] \, dx \text{ or } \int_0^\infty P[S > x] \, dx.$$

Theorem. Let $S$ be a discrete random variable with state space $\{0, 1, 2, 3, \ldots \}$. Then $E[S] = \sum_{n=0}^\infty P[S > n]$.

Proof. Picture $np(n)$ as $n \ p(n)$’s stacked on top of each other.
Cars arrive at a toll plaza with \( s \) toll booths, get serviced and leave. Assume arrival and service times are independent with an exponential distribution with arrival rate \( \lambda \) and service rate \( \mu \).

Since the time between arrivals is exponential, no matter what time \( t \) we pick, the time to the next arrival has the same expected value (the process doesn’t remember how much time has already been spent waiting).

At any time \( t \), the population size \( X_t \) is the total number of cars in the plaza (either waiting or being served) at time \( t \).

On an arrival (birth), \( X_t \) increases by 1.
On the completion of a service (death) \( X_t \) decreases by 1.

For a population of size \( j \), the birth rate is the arrival rate: \( \lambda_j = \lambda \).

Death rates are harder.
If \( X_t = 0 \), deaths are not possible and the death rate is \( \mu = 0 \).
**Case** $1 \leq j \leq s$. Thus all $j$ cars on the plaza are currently being served. The next change will either be an arrival (birth rate $= \lambda$) or service completion for one of the $j$ cars being served (death rate $= j\mu$).

The overall rate of change (arrivals + service completions) $= \lambda + j\mu$. Probability the next change is an arrival (birth) $= \lambda / (\lambda + j\mu)$. Probability the next change is a death (service completion) is $j\mu / (\lambda + j\mu)$.

**Case** $s < j$. In this case only $s$ of the $j$ are being serviced and the rest wait in the queue. The next change will either be an arrival (birth rate $= \lambda_j = \lambda$) or a service completion for one of the $s$ cars being served (death rate $= \mu_j = s\mu$).

The overall rate of change is $\lambda + s\mu$. The probability the next change is birth is $\lambda / (\lambda + s\mu)$. The probability the next change is a death is $s\mu / (\lambda + s\mu)$. The birthrates are $\lambda_j = \lambda$.

Combining the two cases gives: The birth rates are $\lambda_j = \lambda$. The death rates are $\mu_j = j\mu$ if $0 \leq j \leq s$, $\mu_j = s\mu$ if $j \geq s$. 
Here are Kolmogorov’s equations for the long-term probability \( p_j \) that the number of cars on the plaza is \( j \):

\[
p_0 = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} + \ldots}
\]

The denominator is

\[
1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} + \ldots
\]

\[
+ \left( \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \ldots \frac{\lambda_{s-1}}{\mu_s} \right) + \left( \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \ldots \frac{\lambda_s}{\mu_{s+1}} \right) + \ldots
\]

\[
= 1 + \frac{\lambda}{1 \mu} + \frac{\lambda}{1 \mu} \frac{\lambda}{2 \mu} + \frac{\lambda}{1 \mu} \frac{\lambda}{2 \mu} \frac{\lambda}{3 \mu} + \ldots + \frac{\lambda^s}{s! \mu^s}
\]

\[
+ \frac{\lambda^{s+1}}{s! \mu^s (s \mu)} + \frac{\lambda^{s+2}}{s! \mu^s (s \mu)^2} + \ldots + \frac{\lambda^{s+k}}{s! \mu^s (s \mu)^k} + \ldots
\]

\[
= \sum_{j=0}^{s} \frac{\lambda^j}{j! \mu^j} + \sum_{k=1}^{\infty} \frac{\lambda^{s+k}}{s! s^k \mu^{s+k}}
\]

\[
= 1/\left[ \sum_{j=0}^{s} \left( \frac{\lambda}{\mu} \right)^j \frac{1}{j!} + \left( \frac{\lambda}{\mu} \right)^s \frac{1}{s!} \sum_{k=1}^{\infty} \left( \frac{\lambda}{s \mu} \right)^k \right]
\]

The last sum is \( \frac{\lambda}{s \mu} / [1 - \frac{\lambda}{s \mu}] = \frac{\lambda}{s \mu - \lambda} \). Hence

\[
p_0 = 1/\left[ \sum_{j=0}^{s} \left( \frac{\lambda}{\mu} \right)^j \frac{1}{j!} + \left( \frac{\lambda}{\mu} \right)^s \frac{\lambda}{s! (s \mu - \lambda)} \right]
\]
\[ p_j = \frac{\lambda_0}{\mu} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{j-1}}{\mu_j} p_0 \]

\[ = \left( \frac{\lambda}{\mu} \right)^j \frac{1}{j!} p_0 \text{ if } j \leq s. \]

\[ p_j = \left( \frac{\lambda}{\mu} \right)^j \frac{1}{s! s^{j-s}} p_0 \text{ if } s < j. \]

**Theorem.** To keep the queue from becoming infinitely large we must have \( \lambda < \mu s. \)

**Proof.** To keep the queue length bounded, large size queues must have small probabilities. As the queue size \( j \) goes to \( \infty \), the probability \( p_j \) must go to 0.

\[ \lim_{j \to \infty} p_j = \lim_{j \to \infty} \left( \frac{\lambda}{\mu} \right)^j \frac{1}{s! s^{j-s}} p_0 \]

\[ = \lim_{j \to \infty} \left( \frac{\lambda}{\mu s} \right)^j \frac{s^s}{s!} p_0 = \frac{s^s}{s!} p_0 \lim_{j \to \infty} \left( \frac{\lambda}{\mu s} \right)^j. \]

This is 0 iff \( \frac{\lambda}{\mu s} < 1 \) iff \( \lambda < \mu s. \)

This just says the obvious: the birthrate (number of arrivals) must be less than the death rate (number which can be served).