Two-person zero-sum games

A game is played for money between two players: player I and player II. It is a zero-sum game iff the total amount the two players have doesn’t change iff what one player wins, the other loses iff the amount I wins = the negative of the amount II wins. The amount I wins in one play of the game (the amount player II loses) = the payoff.

Each player has a finite number of ways or strategies for playing the game. The payoff matrix lists I’s strategies in the left column and II’s strategies in the top row; the entries in the matrix are the payoffs (I’s wins, II’s losses) which result when I and II play according to their strategies.

- Suppose I and II choose letters from \{a, b, c\}. If the letters equal, both get $0. Otherwise the one with the alphabetically last letter gets $1, the other loses $1.

  The payoff matrix is:

<table>
<thead>
<tr>
<th>II</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>Expected win for I</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

  I tries to maximize the payoff which is how much he wins.

  II tries to minimize the payoff which is how much he loses.

  If player I consistently chooses a and II plays intelligently, what can I expect to win? Likewise if I consistently chooses b or c.

- Now suppose the matrix is

<table>
<thead>
<tr>
<th>II</th>
<th>x</th>
<th>y</th>
<th>Expected win for I</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>9</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>8</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

  I has strategies a, b; II has strategies x, y. If I consistently plays strategy a or b what can he expect to win?

  **Lemma.** The amount I can expect to win by consistently playing the strategy for a given row is the minimum payoff for that row.

  **Maximin Theorem.** The strategy for I with the maximum expected win is the strategy whose row has the maximum minimum. This is the maximin strategy for I.

For I, playing the maximin strategy is better than playing any other single strategy but playing a random mixed strategy may have a higher average payoff.

If a and b are strategies, then \(.3a + .7b\) is the mixed strategy where in each play of the game one of the two strategies is randomly chosen with a chosen 30% of the time and b 70% of the time.

- I and II choose heads H or tails T. If they choose the same side, I wins $1, otherwise II wins $1.

<table>
<thead>
<tr>
<th>II</th>
<th>H</th>
<th>T</th>
<th>Expected win for I</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>.5H+.5T</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

  If I consistently plays H or T, he can expect to lose $1. Suppose I plays according to the mixed strategy \(.5H+.5T\), i.e., he chooses H, T randomly, each with probability 1/2.

  If II chooses H,

  1/2 of the time I wins $1 and 1/2 of time I loses $1.

  On average I wins \(.5(1) + .5(-1) = 0.\)

  If II chooses T, I can likewise expect to win $0.

  This mixed strategy for I gives $0 on average; either single strategy gives $-1.

  **Lemma.** If f and g are intersecting functions, one increasing, one decreasing, then \(p\) maximizes the minimum of \(f(p)\) and \(g(p)\) iff \(f(p) = g(p)\) iff \(p\) minimizes the maximum of \(f(p)\) and \(g(p)\).

  \[
  \max_{z=y} z = y \\
  y \leq f(p) \\
  y \leq g(p)
  \]
Suppose player $I$ chooses strategy $A$ with probability $p$ and chooses strategy $B$ with probability $1-p$.

If $II$ plays strategy $X$, then he loses $6$ with probability $p$ and loses $2$ with probability $1-p$. His average loss is $p \cdot 6 + (1-p) \cdot 2 = 4p+2$.

If $II$ plays $Y$, his average loss is $p \cdot 4 + (1-p) \cdot 8 = 8-4p$.

To minimize his losses, $II$ selects the minimum of $4p+2$ and $8-4p$.

To maximize his winnings, $I$ should choose the $p$ which maximizes the minimum of $4p+2$ and $8-4p$.

By the Lemma above, this is the $p$ such that $4p+2 = 8-4p$. Thus $8p = 6$ and hence $p = 3/4$.

\textbf{Answer.} Player $I$ should randomly choose strategy $A$ with probability $3/4$ and choose strategy $B$ with probability $1/4$. His expected winnings are $\min(4p+2, 8-4p)$, by choice of $p$, $= 4p+2 = 4(3/4)+2 = 5$.

Note, $5$ is more than the $4$ or $2$ he gets by playing either $A$ or $B$ alone.

Now consider games from $II$'s point of view (dual to $I$'s point of view). The payoff matrix gives $I$'s winnings. These are $II$'s losses. Assume $I$ plays intelligently.

\textbf{Lemma.} The amount $II$ can expect to lose by consistently playing a single strategy for a given column is the maximum payoff for that column.

\textbf{Theorem.} The single strategy for $II$ with the minimum expected loss is the strategy whose column has the minimum maximum. This strategy is called $II$'s \textit{minimax} strategy.

\textbf{Definition.} A game is \textit{strictly determined} if the entry in the maximin row and the minimax column = the payoff for $I$'s maximin strategy = the payoff for $II$'s minimax strategy. This entry is the \textit{saddle or equilibrium point} of the game.

\textbf{Theorem.} If there is an equilibrium point, then both players can optimize their earnings by always playing their simple (unmixed) maximin and minimax strategies.

The equilibrium point is stable in the sense that neither player has any incentive to change his strategy. If a game is not strictly determined, some player will do better with a mixed strategy using random moves.
This is a smaller loss then the $6 or $8 II loses by playing either X or Y alone. Note that his expected $5 loss is exactly the $5 expected win for I’s optimal strategy.

**Theorem.** When I and II play according to their optimal mixed strategies, I’s expected winnings equals II’s expected losses.

**Problem.** Analyze the game “stone, paper, scissors”.

<table>
<thead>
<tr>
<th></th>
<th>Stone</th>
<th>Paper</th>
<th>Scissors</th>
<th>Expected win for I</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stone</td>
<td>p</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Paper</td>
<td>q</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Scissors</td>
<td>r</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$I$’s optimal strategy is to find the $p$, $q$, $r$ which maximizes his expected winnings

\[ w = \min(q - r, r - p, p - q) \text{ with } p + q + r = 1. \]

This gives the expected answer: $m = 0, p = q = r = 1/3$. 

\[ m = \begin{cases} 
0 & \text{if } m = 0, \\
q & \text{if } m = q, \\
r & \text{if } m = r.
\end{cases} \]