Let \( F_k(i, j) = P[PT(j) = k | X_0 = i] \) be the probability that \( PT(j) = k \) steps from the initial state \( i \).

**Theorem.** \( F_0(i, j) = 1 \) iff \( i = j \),
\[
F_1(i, j) = T(i, j), \quad F_k(i, j) = \sum_{x \in S \setminus \{j\}} T(i, x) F_{k-1}(x, j).
\]

**Definition.** A state \( x \) is **absorbing** if you can’t check out. If, starting from \( x \), there is a nonzero probability it is visited only finitely often, it is **transient**. If, once visited, it is visited, with probability 1, infinitely often, it is **recurrent**.

**Definition.** A set of states is **closed** iff no arrow leads from a state in the set to a state outside the set iff one can not leave the set. A set of states is **irreducible** iff it is closed and no proper subset is closed.

**Theorem.** Starting from a state \( j \), the expected number of return visits to \( j \) equals \( \sum_{n=1}^{\infty} T^n(j, j) \).

**Theorem.** Suppose \( i \) and \( j \) are states. Then

(a) If \( j \) is not reachable from \( i \), then for all \( n \), \( T^n(i, j) = 0 \).

(b) If \( j \) is transient, then \( \lim_{n \to \infty} T^n(i, j) = 0 \).

Let \( \pi_j = \lim_{n \to \infty} T^n(i, j) \). This is the “long-term” probability of being at \( j \). Let \( M_j \) be the average time between visits of \( j \). For a given chain \( \omega \) of states, let \( N_n(j, \omega) \) be the total number of visits to \( j \) up to time and including \( n \). Thus \( N_n(j, \omega)/n \) is the percentage of time during \([0, n]\) spent in state \( j \).

**Theorem.** If \( i \) and \( j \) are in the same recurrence class and the class matrix is regular, then
\[
\pi_j = \lim_{n \to \infty} T^n(i, j) = T^\infty(i, j).
\]

With probability 1, \( \pi_j = \lim_{n \to \infty} N_n(j, \omega)/n \)
\[
\pi_j = 1/M_j
\]

**Theorem.** If \( T \) is a regular matrix for an irreducible space or a submatrix for a recurrence class, then:

- The rows of \( T^\infty \) are identical. Let \( \pi \) be the common row. Its \( j \)th entry is the proportion \( \pi_j \) of time in state \( j \).
- The entries of a row \( \pi \) are probabilities that total to 1.
- For each row \( \pi \), \( \pi \times T = \pi \).

\( \pi \) is the **steady-state distribution**. The equations \( \pi = \pi T \) and \( \sum \pi_j = 1 \) are the **steady-state equations**.

**Theorem.** \( T^\infty(i, j) = f_{ij} \cdot \pi_j \).

**Theorem.** \( f_{ij} = \sum_{i' \text{ transient}} T(i, i') f_{ij} + \sum_{j' = \text{transient}} T(i, j') \).

\( f_{ij} \) also occurs in the first sum on the left since \( i \) is one of the transient \( i' \).

Events have a Poisson distribution iff the times between the randomly occurring events are exponentially distributed with density function \( \lambda e^{-\lambda t} \) where \( t \) is time and \( \lambda \) is the rate. The mean time of the distribution is \( 1/\lambda \).

\( N_t \) counts the number of arrivals, \( T_n \) is the time of the \( n \)th arrival.
\[
N_t < n \Leftrightarrow T_n > t, \quad n \leq N_t \Leftrightarrow T_n \leq t, \quad N_t = n \Leftrightarrow t \in [T_n, T_{n+1}).
\]

**Theorem.** For a Poisson process \( N_t \) with arrival times \( T_n \):

(a) For a given \( t \), \( P[N_t = n] = e^{-\lambda t} \left( \frac{\lambda t}{n!} \right)^n \), \( E[N_t] = \lambda t \).

(b) For a given \( n \),
\[
P[N_t < n] = P[T_n > t] = \sum_{k=0}^{n-1} e^{-\lambda t} \left( \frac{\lambda t}{k!} \right)^k, \quad E[T_n] = n/\lambda.
\]

**Theorem.** The number of events between time \( t \) and time \( t+s \) is \( N_{t+s} - N_t \). This number is independent of the previous history \( N_u : u \leq t \). It is independent of \( t \).
\[
E(N_{t+s} - N_t) = E(N_s - N_0) = E(N_s - 0) = E(N_s).
\]

**Theorem.** Given that there are \( n \) arrivals during the time period \([0, t]\), the uniform (it depends on \( t \) but not \( t_1, ..., t_n \)) joint density function for \( T_1, T_2, ..., T_n \) is \( f(t_1, t_2, ..., t_n) = n! t^n \) for \( 0 < t_1 < t_2 < ... < t_n < t \).

\[
\int_0^{t_{\infty}} \cdots \int_0^{t_{\infty}} c = \int_0^t \cdots \int_0^t 0 \, dx \, dy \, dz.
\]

**Definition.** Let \( p_{ij}(t) = P[N_t = j | N_0 = i] \). Let \( \bar{p}_j = \lim_{t \to \infty} p_{ij}(t) \).

**Kolmogorov Equations**

**Definition.** Let \( p_{ij}(t) = P[N_t = j | N_0 = i] \). Let \( \bar{p}_j = \lim_{t \to \infty} p_{ij}(t) \). Here \( N_t \) is the population at time \( t \)

As was the case for \( \pi \) and regular Markov chains, under rather general regularity conditions, the limit exists and \( \bar{p}_j = \lim_{t \to \infty} p_{ij}(t) \).

**Kolmogorov’s Theorem.** Assuming the population converges to a stable distribution independent of the initial state, we have
\[
p_0 = \lim_{t \to \infty} [N_t = 0 | N_0 = i] = 1/(1 + \sum_{j=1}^n \frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_j}{\mu_j})
\]

and
\[
p_j = \lim_{t \to \infty} [N_t = j | N_0 = i] = (\lambda_0 \lambda_1 \cdots \lambda_{j-1}) \mu_1 \mu_2 \cdots \mu_j p_0
\]

**Poisson Arrival Queues**

Suppose the queue has type Poisson/\( G(x) / \infty \). That is, the arrivals are Poisson with some rate \( \lambda \). There are infinitely many lines a customer can choose from. The service times have a cumulative probability function \( G(x) \). Then the expected number of customers in the system at time \( t \) is
\[
E[X_t] = p \lambda t = \lambda t (1/t) \int_0^\infty (1 - G(x)) \, dx = \lambda \int_0^\infty (1 - G(x)) \, dx
\]

If the service times are exponentially distributed with rate \( \mu \). Then \( E[X_t] = \lambda \int_0^\infty e^{-\mu x} \, dx = (\lambda/\mu) (1 - e^{-\mu t}) \).

As time goes to \( \infty \), the expected number being served goes to \( \lambda/\mu \).