Math 241 Final Exam, Spring 2013

Name:

Section number:

Instructor:

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<td><strong>Total:</strong></td>
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Read all of the following information before starting the exam.

- Electronic devices (calculators, cell phones, computers), books and notes are not allowed.
- Show all work clearly. You may lose points if we cannot see how you arrived at your solution.
- You do not have to simplify your arithmetic. But be aware that if your answer looks like you need a calculator, you are probably doing it wrong.
- Box or otherwise clearly indicate your final answers.
- This test has 12 pages total including this cover sheet and is worth 140 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!
1. (5 points) Compute the following limit

\[ \lim_{t \to 4} \left( \frac{t - 4}{\sqrt{t} - 2} \right) \]

2. (5 points) Compute the following limit, making sure to justify all steps.

\[ \lim_{x \to \infty} \frac{3x \sin(x^2) + 1}{x^2 + 4} \]
3. The graph of the function \( y = f(x) \) is given below:

You will not receive full credit without an explanation at each point.

(a) (4 points) At which points in \([-5, 5]\) does the limit of the function not exist? Explain why.

(b) (4 points) At which points in \([-5, 5]\) is the function discontinuous? Explain why.

(c) (4 points) At which points in \([-5, 5]\) is the function not differentiable? Explain why.
4. (10 points) Using the definition of derivative, NOT differentiation rules, find $f'(2)$ if \[ f(x) = \frac{1}{x+1}. \]
5. Compute the following integrals
   (a) (5 points) \( \int \sin(3x + 1) \, dx \)
   
   (b) (5 points) \( \int_{-1}^{2} t^3 - t \, dt \)
   
   (c) (7 points) \( \int_{0}^{1} x(x - 1)^{2013} \, dx \)
6. (15 points) This question refers to the curve defined by

\[ 2x^3 + xy = y^2. \]

(a) Find \(dy/dx\) at the point (1, 2).

(b) Find the tangent line to the curve at (1, 2).

(c) Using the tangent line, estimate the value of \(y\) at \(x = 1.2\). [You do not have to simplify the arithmetic.]
7. (6 points) Does the equation $x^3 - 4x^2 + 2 = 1$ have a solution on the interval $[-1, 1]$? Make sure you justify your answer, and clearly state any theorems that you are using.
8. (12 points) Bob wants to use 12 square inches of metal to make a cylindrical can of height $h$ and radius $r$. Bob just wants the sides and bottom of the can to be included, but not the top. What is the maximum volume of the can that Bob can make?

_Hint: the volume of a cylinder as given is $V = \pi r^2 h$. _
9. A 13 foot ladder is leaning against a wall when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

(a) (4 points) How fast is the top of the ladder sliding down the wall at that time?

(b) (4 points) At what rate is the area of the triangle formed by the ladder, wall and ground changing at that time?

(c) (4 points) At what rate is the angle $\theta$ between the ladder and the ground changing at that time?
10. Let $f(x) = x^4 - 4x^3 + 10$. Then $f'(x) = 4x^2(x - 3)$, $f''(x) = 12x(x - 2)$.

(a) (4 points) Find the intervals where $f$ is increasing, and those where it is decreasing.

(b) (4 points) Find the local and global maxima and minima. Give both the values, and where they are attained.

(c) (4 points) Find the intervals where $f$ is concave up, and those where it is concave down.

(d) (2 points) Find the inflection points of $f$. 
11. Below is a plot of a function $y = f(x)$. It is linear on the intervals $[-5, -2]$, $[2, 3]$, and $[3, 5]$, and a radius 2 semicircle on the interval $[-2, 2]$:

Define $F(x) = \int_{0}^{x} f(t) \, dt$ for $x$ in the interval $[-5, 5]$.

(a) (8 points) Evaluate the following

\[ F(4) = \]
\[ F'(4) = \]
\[ F''(4) = \]
\[ F(-2) = \]

(b) (4 points) On what interval(s) is the function $F$ increasing?

(c) (5 points) Another function $G(x)$ is defined as $G(x) = \int_{0}^{x} f(t) \, dt$. Find $G'(2)$. 

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12. Let \( f(x) = x \) and \( g(x) = x^3 \). Let \( R \) be the region between the graphs of \( f \) and \( g \) as \( x \) ranges over the interval \([0, 1]\).

(a) (5 points) Find the volume of the region arrived at by rotating \( R \) about the x-axis.

(b) (5 points) Set up, but do not integrate, an integral for the volume when \( R \) is rotated about the axis \( y = 1 \).

13. (5 points) Leslie would like to approximate the area under the following curve on the interval \([-1, 1]\) using a Riemann sum and intervals of length 0.50. Sketch the areas she should compute in order to guarantee that the approximation is an overestimate.
Solutions

These are possible guidelines for grading along with correct solutions. Any correct solution deserves credit; it does not have to be this solution.

1. \[
\lim_{t\to4} \left( \frac{t - 4}{\sqrt{t} - 2} \right) = \left( \lim_{t\to4} \frac{t - 4}{\sqrt{t} - 2} \right) = \lim_{t\to4} \left( \frac{t - 4}{\sqrt{t} - 2} \right) = \left( \lim_{t\to4} \sqrt{t} + 2 \right) = 4.
\]

2 points - using conjugate.
1 point - takes limit.
2 points - algebraic accuracy.

TOTAL: 5 points.

2. As \(-1 \leq \sin(x^2) \leq 1\) for any \(x\), we have
\[
\frac{-3x + 1}{x^2 + 4} \leq \frac{3x \sin x^2 + 1}{x^2 + 4} \leq \frac{3x + 1}{x^2 + 4}.
\]

Taking the limits of the upper and lower bounds gives
\[
\lim_{x\to\infty} \frac{-3x + 1}{x^2 + 4} = \lim_{x\to\infty} \frac{3x + 1}{x^2 + 4} = 0
\]
(as the degree of the denominator is higher than that of the numerator). Hence, by the sandwich theorem,
\[
\lim_{x\to\infty} \frac{3x \sin x^2 + 1}{x^2 + 4} = 0.
\]

2 points - estimates of sin term (somehow)
2 points - find limit(s) of function(s) without sin term.
1 point - final answer correct.

TOTAL: 5 points.

3. (a) At \(x = -3\) since the function becomes unbounded. At \(x = -1\) since the limit from the left and the limit from the right are not the same.

2 points - one for each point.
2 points - one for each explanation.

TOTAL: 4 points.

(b) At \(x = -1\) since the limit of the function does not exist there. (At \(x = -3\) the function is undefined – ignore this if mentioned.) Also at \(x = 2\) since \(\lim_{x\to2} f(x) = 3\) which is not equal to \(f(2) = -3\).

1 point - previous points.
1 point - new point.
2 points - explanations.

TOTAL: 3 points.
(c) At \( x = -1,2 \) since the function is not continuous there. (Again ignore a mention of \( x = -3 \).) Also at \( x = 0 \) since the limit of the derivative as \( 0^− \) is \( -1 \) and the derivative at \( 0^+ \) is \( +1 \).

1 point - previous points.
1 point - new point.
2 points - explanations.

**TOTAL:** 3 points.

4. Using the definition of derivative

\[
f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}
\]

(the other standard form of the definition is fine too, of course). Hence

\[
f'(2) = \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{3 + h} - \frac{1}{3} \right) = \lim_{h \to 0} \frac{1}{h} \left( -\frac{h}{3(3 + h)} \right) = \lim_{h \to 0} \left( \frac{-1}{3(3 + h)} \right) = -\frac{1}{9}
\]

4 points - definition of derivative.
1 point - substition of \( x = 2 \) correctly.
5 points - accurate algebra.
**TOTAL:** 10 points.

5. (a) u-substitution with \( u = 3x + 1 \), also needs to know integral of trig function

\[
\int \sin(3x + 1) \, dx = -\frac{1}{3} \cos(3x + 1) + C
\]

3 points - more than one minor mistake; OR
4 points - one minor mistake; OR
5 points - correct.

**TOTAL:** 5 points.

(b)

\[
\int_{-1}^{2} t^3 - t \, dt = \left. \frac{t^4}{4} - \frac{t^2}{2} \right|_{-1}^{2} = 4 - 2 - \left( \frac{1}{4} - \frac{1}{2} \right) = \frac{9}{4}
\]

3 points - more than one minor mistake; OR
4 points - one minor mistake; OR
5 points - correct.

**TOTAL:** 5 points.
(c) u-substitution with \( u = x - 1 \)

\[
\int_0^1 x(x-1)^{2013} \, dx = \int_{-1}^0 (u+1)u^{2013} \, du
\]

\[
= \int_{-1}^0 u^{2014} + u^{2013} \, du
\]

\[
= \left. \frac{1}{2015}u^{2015} + \frac{1}{2014}u^{2014} \right|_{-1}^0
\]

\[
= \frac{1}{2015}(-1)^{2015} - \frac{1}{2014}(-1)^{2014}
\]

\[
= \frac{1}{2015} - \frac{1}{2014}
\]

-1 per minor mistake, -2 per major mistake (eg. using wrong limits at the end, as opposed to intermediate steps)
7 points - correct.

**TOTAL:** 7 points.

6. Implicitly differentiating gives

\[
6x^2 + y + xy' = 2yy'
\]

Thus

\[
dy/dx = \frac{6x^2 + y}{2y - x}.
\]

Evaluating at (1, 2) gives

\[
6 + 2 + y' = 4y' \quad \Rightarrow \quad 3y' = 8 \quad \Rightarrow \quad y' = \frac{8}{3}
\]

The equation for the line is therefore:

\[
y - 2 = \frac{8}{3}(x - 1) \quad \Rightarrow \quad y = \frac{8}{3}x - \frac{2}{3}
\]

\[
y(1.2) \approx y(1) + y'(1)(.2) = 2 + \frac{8}{3}(.2)
\]

5 points for each part, allowing the student to carry his answer to the next part unless he makes the problem trivial.

**TOTAL:** 15 points.

7. The intermediate value theorem states that if \( f : [a, b] \rightarrow \mathbb{R} \) is a continuous function on a closed bounded interval, and \( y \) is any point between (or equal to one of) \( f(a) \) and \( f(b) \) there exists \( c \in [a, b] \) with \( f(c) = y \).

[This is one option for a solution.] Consider the function \( f(x) = x^3 - 4x^2 + 2 \), which is a polynomial, so continuous everywhere. In the notation of the theorem, let \( a = 0, b = 1 \).

Then

\[
f(a) = 2, \quad f(b) = -1;
\]
as 1 is between 2 and −1, the intermediate value theorem implies that there exists c in [0, 1] (so in particular, in [−1, 1]) such that \( f(c) = 1 \), i.e. the given equation has a solution in [−1, 1].

3 points - correct statement of intermediate value theorem (1 for continuity assumption, one for connectedness (interval) assumption, 1 for conclusion).
1 point - check two points in [−1, 1] (even if just endpoints, which is no use).
1 point - check assumptions of theorem (don’t be too rigid - a mention of continuity somewhere should be sufficient, not a statement that polynomials are continuous).
1 point - correct conclusion.
TOTAL: 6 points.

8. The surface area of the can is

\[
A = \pi r^2 + 2\pi rh \text{ (inches)}^2.
\]

(4 points: 2 for each term). Setting this equal to 12 gives

\[
h = \frac{12 - \pi r^2}{2\pi r},
\]

and substituting into the volume equation gives

\[
V = \frac{1}{2}(12r - \pi r^3).
\]

(3 points - getting to an equation with one variable).
Differentiating with respect to \( r \) and setting equal to zero gives

\[
0 = \frac{1}{2}(12 - 3\pi r^2)
\]

(4 points - 3 points for knowing to differentiate and set to zero; 1 point for doing it accurately).
Solving gives

\[
12 = 3\pi r^2 \Rightarrow \pi r^2 = 4 \Rightarrow r = \frac{2}{\sqrt{\pi}} \text{ inches}
\]

(1 point) Finally, substituting this into the equation for \( h \) gives

\[
h = \frac{12 - 4}{4\sqrt{\pi}} = \frac{2}{\sqrt{\pi}} \text{ inches}.
\]

TOTAL: 12 points.

9. (a) (1 point) Set up the equation as \( x^2 + y^2 = 13^2 \)
(2 points) Correctly differentiate \( 2xx' + 2yy' = 0 \).
(1 point) Solve for \( y' = -\frac{12.5}{5} = -12 \text{ ft/sec} \) (units not required).

(b) (1 point) Set up the equation \( A = \frac{1}{2}xy \)
(2 points) Differentiate \( A' = \frac{1}{2}(x'y + xy') \).
(1 point) Substitute \( A' = \frac{1}{2}(5 \cdot 5 + 12 \cdot (-12)) = -59.5 \text{ ft}^2/\text{sec} \) (units and simplification not required)
(c) (1 point) Set up equation \( \tan(\theta) = \frac{y}{x} \).
(2 points) Differentiate
\[
\sec^2(\theta)\theta' = \frac{y'x - yx'}{x^2}
\]
(1 point) Substitute and solve \( \theta' = \frac{(-12)12-5.5}{12^2} (\frac{12}{13})^2 = -1 \text{ radian/sec.} \)

10. (a) We have
\[ f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3). \]
This is zero at 0 and 3. Checking the intervals involved gives that \( f \) is decreasing on \((-\infty, 0]\), decreasing on \([0, 3]\) (so decreasing on all of \((-\infty, 3]\)) and increasing on \([3, \infty)\).
1 point - derivative.
1 point - find zeros.
1 point - know to check signs on intervals.
1 point - correct solution.
(Note: give full credit if corresponding open intervals are given in answers).
**TOTAL**: 4 points.

(b) From (a), the zeros are 0 and 3, and 0 is neither a local maximum nor a local minimum, while 3 is a local minimum.
As \( f \) tends to infinity as \( x \to \pm\infty \), \( f \) has no global maximum. It has one local minimum at \( x = 3 \) with value \( f(3) = -17 \), which is also its global minimum.
1 point - read off zeros and local mins / maxes from (a).
1 point - no global max.
1 point - local minimum at right point.
1 point - no global max.
**TOTAL**: 4 points.

(c) The second derivative of \( f \) is
\[ f''(x) = 12x^2 - 24x = 12x(x - 2). \]
This is zero at \( x = 0 \) and \( x = 2 \). Checking the intervals involved shows that \( f \) is concave up on \((-\infty, 0]\), concave down on \([0, 2]\) and concave up on \([2, \infty)\).
1 point - second derivative.
1 point - find zeros.
1 point - know to check signs on intervals.
1 point - correct solution.
(Note: give full credit if corresponding open intervals are given in answers).
**TOTAL**: 4 points.

(d) From part (c), the inflection points are 0 and 2.
1 point each
**TOTAL**: 2 points.

11. (a)
\[
\begin{align*}
F(4) &= \pi - 1.5 - 3 \\
F'(4) &= -3 \\
F''(4) &= 0 \\
F(-2) &= -\pi.
\end{align*}
\]
(b) Increasing on $[-5, 2]$. 

**TOTAL**: 4 points (2 points for wrong answer $[0, 2]$).

(c) 

$$G'(x) = 2xf(x^2) \implies G'(2) = 4(-3) = -12$$

**TOTAL**: 5 points.

12. (a) 

$$\int_{0}^{1} (\pi x^2 - \pi x^6) \, dx = \pi \left( \frac{x^3}{3} - \frac{x^7}{7} \right) \bigg|_{0}^{1} = \frac{4}{21} \pi$$

5 points - 1 for limits, 1 for $\pi$, 1 for proper difference of terms, 1 for integration, 1 for answer

(b) Using the ‘shell’ method, the integral we want is 

$$\int_{0}^{1} \pi((1 - x^3)^2 - (1 - x)^2) \, dx.$$ 

5 points - set up integral correctly (1 for bounds, 1 for $\pi$, 2 for terms $(1 - x^3)^2$ and $(1 - x)^2$, 1 for minus sign in right place).

**TOTAL**: 10 points.

13. Since the graph is increasing (and concave up) the Riemann sum will overestimate the area under the curve if the right-endpoint rule is used. This should be drawn.

**TOTAL**: 5 points.