A1. Find polynomials \( f(x), g(x), \) and \( h(x) \), if they exist, such that for all \( x \),

\[
|f(x)| - |g(x)| + h(x) = \begin{cases} 
-1 & \text{if } x < -1 \\
3x + 2 & \text{if } 1 \leq x \leq 0 \\
-2x + 2 & \text{if } x > 0 
\end{cases}
\]

Solution. Let \( h(x) = ax + b \). Note that there are corners in the graph of this function at \( x = -1 \) and \( x = 0 \). Thus, we expect \( f(x) = c(x + 1) \) and \( g(x) = dx \) for some positive constants \( c \) and \( d \). For \( x < -1 \), we have

\[
-1 = |f(x)| - |g(x)| + h(x) = -c(x + 1) + dx + ax + b \\
= (d - c + a)x + b - c
\]

Thus, \( b - c = -1 \) and \( d - c + a = 0 \). For \( x > 0 \),

\[
-2x + 2 = |f(x)| - |g(x)| + h(x) = c(x + 1) - dx + ax + b \\
= (c - d + a)x + b + c
\]

Thus, \( b + c = 2 \) and \( c - d + a = -2 \). Now

\[
b - c = -1 \text{ and } b + c = 2 \Rightarrow b = \frac{1}{2}, c = \frac{3}{2} \\
d - c + a = 0 \text{ and } c - d + a = -2 \Rightarrow a = -1, d = c - a = \frac{5}{2}
\]

Thus,

\[
f(x) = \frac{3}{2}(x + 1), \quad g(x) = \frac{5}{2}x, \quad h(x) = -x + \frac{1}{2}
\]

is correct for \( x < -1 \) and \( x > 0 \). Continuity gives \( 3x + 2 \) for \( -1 \leq x \leq 0 \).

A2. Let \( p(x) \) be a polynomial that is nonnegative for all real \( x \). Prove that for some \( k \), there are polynomials \( f_1(x), \ldots, f_k(x) \) such that

\[
p(x) = \sum_{j=1}^{k} (f_j(x))^2.
\]

Solution. Clearly \( p(x) \) has real coefficients, and we may assume \( p(x) \) is monic. Let \( p(x) = r(x) c(x) \) where the roots of \( r(x) \) are all real and the roots of \( c(x) \) are not real. We have

\[
r(x) = \prod_j (x - r_j)^2 = \left( \prod_j (x - r_j) \right)^2 = s(x)^2 \text{ and } \\
c(x) = \prod_k \left( (x - a_k)^2 + b_k^2 \right)
\]
Note that when \( c(x) \) is multiplied out it is a sum of squares of polynomials, say \( c(x) = \sum_{m=1}^{K} g_m(x)^2 \); and

\[
p(x) = s(x)^2 c(x) = \sum_{m=1}^{K} (s(x) g_m(x))^2,
\]
as required.

**A3.** Consider the power series expansion

\[
\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.
\]
Prove that, for each integer \( n \geq 0 \), there is an integer \( m \) such that

\[
a_n^2 + a_{n+1}^2 = a_m.
\]

**Solution.** We do a partial fractions decomposition:

\[
\frac{1}{1 - 2x - x^2} = \frac{1}{2 - (x + 1)^2}
\]

\[
= \frac{1}{(\sqrt{2} - (x + 1)) (\sqrt{2} + (x + 1))}
\]

\[
= \frac{A}{\sqrt{2} - 1 - x} + \frac{B}{\sqrt{2} + 1 + x}
\]

\[
\Rightarrow 1 = A (\sqrt{2} + 1 + x) + B (\sqrt{2} - 1 - x)
\]

\[
\Rightarrow A = B = \frac{1}{2\sqrt{2}}
\]

Thus,

\[
\frac{1}{1 - 2x - x^2} = \frac{1}{2\sqrt{2}} \left( \frac{1}{\sqrt{2} - 1} \left( 1 - \frac{x}{\sqrt{2} - 1} \right) + \frac{1}{\sqrt{2} + 1} \left( 1 + \frac{x}{\sqrt{2} + 1} \right) \right)
\]

\[
= \frac{1}{2\sqrt{2}} \left( \frac{1}{\sqrt{2} - 1} \left( \sum_{n=0}^{\infty} \frac{x^n}{(\sqrt{2} - 1)^n} \right) + \frac{1}{\sqrt{2} + 1} \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(\sqrt{2} + 1)^n} \right) \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{2\sqrt{2}} \left( \frac{1}{(\sqrt{2} - 1)^{n+1}} + \frac{(-1)^n}{(\sqrt{2} + 1)^{n+1}} \right) x^n
\]

Hence,

\[
a_n = \frac{1}{2\sqrt{2}} \left( \frac{1}{(\sqrt{2} - 1)^{n+1}} + \frac{(-1)^n}{(\sqrt{2} + 1)^{n+1}} \right)
\]
and

\[ a_n^2 + a_{n+1}^2 = \frac{1}{8} \left( \frac{1}{(\sqrt{2} - 1)^{n+1}} + \left(\frac{1}{\sqrt{2} + 1}\right)^{n+1} \right)^2 \]

\[ + \frac{1}{8} \left( \frac{1}{(\sqrt{2} - 1)^{n+2}} + \left(\frac{1}{\sqrt{2} + 1}\right)^{n+2} \right)^2 \]

\[ = \frac{1}{8} \left( \frac{1}{(\sqrt{2} - 1)^{2n+2}} + 2(-1)^n + \frac{1}{(\sqrt{2} + 1)^{2n+2}} \right) \]

\[ + \frac{1}{8} \left( \frac{1}{(\sqrt{2} - 1)^{2n+4}} + 2(-1)^{n+1} + \frac{1}{(\sqrt{2} + 1)^{2n+4}} \right) \]

\[ = \frac{1}{8} \left( \frac{(\sqrt{2} - 1)^2 + 1}{(\sqrt{2} - 1)^{2n+4}} + \frac{(\sqrt{2} + 1)^2 + 1}{(\sqrt{2} + 1)^{2n+4}} \right) \]

\[ = \frac{1}{2\sqrt{2}} \left( \frac{\frac{1}{8}2\sqrt{2}(4 - 2\sqrt{2})}{(\sqrt{2} - 1)^{2n+4}} + \frac{\frac{1}{8}2\sqrt{2}(4 + 2\sqrt{2})}{(\sqrt{2} + 1)^{2n+4}} \right) \]

\[ = \frac{1}{2\sqrt{2}} \left( \frac{1}{(\sqrt{2} - 1)^{2n+3}} + \frac{1}{(\sqrt{2} + 1)^{2n+3}} \right) \]

\[ = a_{2n+2} \]

A4. Sum the series

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2n}{3^mn^3m^3 + m^3n}. \]

Solution. Using the fact that \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \), we have

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2n}{3^mn^3m^3 + m^3n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(3^m/m)(3^m/m + 3^n/n)} \]

\[ = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{(3^m/m)(3^m/m + 3^n/n)} + \frac{1}{(3^n/n)(3^n/n + 3^m/m)} \right) \]

\[ = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(3^m/m)(3^n/n)} \]

\[ = \frac{1}{2} \left( \sum_{m=1}^{\infty} \frac{m}{3^m} \right)^2, \]

where we have also used

\[ \frac{1}{a(a+b)} + \frac{1}{b(a+b)} = \frac{b}{ab(a+b)} + \frac{a}{ba(a+b)} = \frac{1}{ab}. \]
Now
\[
\sum_{m=1}^{\infty} \frac{m}{3^m} = \sum_{m=1}^{\infty} \frac{m}{3^m} (\frac{x}{3})^{m-1} = \frac{d}{dx} \left( \sum_{m=1}^{\infty} \left( \frac{x}{3} \right)^m \right)_{x=3} = \frac{3}{(-3+1)^2} = \frac{3}{4}
\]
Thus, the original sum is then
\[
\frac{1}{2} \left( \frac{3}{4} \right)^2 = \frac{9}{32}.
\]

A5. Prove that there is a constant $C$ such that, if $p(x)$ is a polynomial of degree 1999, then
\[
|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx.
\]

**Solution.** We may assume that $p(0) \neq 0$ and that $p(x)$ is monic. Then
\[
p(x) = \prod_{k=1}^{1999} (x - r_k)
\]
where the $r_k \neq 0$ are the possibly complex roots, ordered by modulus. We have
\[
\frac{|p(0)|}{\int_{-1}^{1} |p(x)| \, dx} = \frac{\prod_{k=1}^{1999} |r_k|}{\int_{-1}^{1} \prod_{k=1}^{1999} |x - r_k| \, dx} = \frac{1}{\int_{-1}^{1} \prod_{k=1}^{1999} |1 - \frac{x}{r_k}| \, dx}
\]
We need to show that as a function of the $r_k$, $\int_{-1}^{1} \prod_{k=1}^{1999} |x - r_k| \, dx$ bounded below. For $|r_k| > 2$, we have
\[
|1 - \frac{x}{r_k}| > 1 - \frac{1}{|r_k|} > \frac{1}{2}
\]
Thus,
\[
\int_{-1}^{1} \prod_{k=1}^{1999} \left| \frac{x}{r_k} - 1 \right| \, dx > \frac{1}{2^{1999-N}} \int_{-1}^{1} \prod_{k=1}^{N} \left| \frac{x}{r_k} - 1 \right| \, dx
\]
where the product is over those $|r_k|$ with $|r_k| \leq 2$. For each $N$,
\[
f_N (r_1, r_2, \cdots, r_N) := \int_{-1}^{1} \prod_{k=1}^{N} \left| \frac{x}{r_k} - 1 \right| \, dx
\]
is a continuous positive function of $(r_1, r_2, \cdots, r_N) \in \{|z| \leq 2\}^N$, a compact set. Thus, $f_N$ has a positive minimum, say $M_N$. Finally,
\[
\int_{-1}^{1} \prod_{k=1}^{1999} \left| \frac{x}{r_k} - 1 \right| \, dx > \min_{1 \leq N \leq 1999} \left\{ \frac{1}{2^{1999-N}} M_N \right\} > 0.
\]
We can take \( C = \min_{1 \leq N \leq 1999} \{ \frac{1}{M_N} \}^{-1} \).

**A6.** The sequence \((a_n)_{n \geq 1}\) is defined by \(a_1 = 1, a_2 = 2, a_3 = 24\), and, for \(n \geq 4\),

\[
a_n = \frac{6a_{n-1}a_{n-3} - 8a_{n-1}a_{n-2}}{a_{n-2}a_{n-3}}.
\]

Show that, for all \(n\), \(a_n\) is an integer multiple of \(n\).

**Solution.**

\[
a_n = \frac{6a_{n-1}a_{n-3} - 8a_{n-1}a_{n-2}}{a_{n-2}a_{n-3}} \Rightarrow \frac{a_n}{a_{n-1}} = \frac{6a_{n-1}a_{n-3} - 8a_{n-2}}{a_{n-2}a_{n-3}} = 6\frac{a_{n-1}}{a_{n-2}} - 8\frac{a_{n-2}}{a_{n-3}}.
\]

Thus, with \(b_n := \frac{a_n}{a_{n-1}}\), we have

\[
b_n = 6b_{n-1} - 8b_{n-2},
\]

Trying \(b_n = r^n\), we get

\[
0 = r^n - 6r^{n-1} + 8r^{n-2} \Rightarrow 0 = (r^2 - 6r + 8) r^{n-2} = (r - 2) (r - 4) r^{n-2}
\]

Hence,

\[
b_n = c_1 2^n + c_2 4^n
\]

and

\[
b_2 = \frac{a_2}{a_1} = 2 \Rightarrow 2 = c_1 2^2 + c_2 4^2 = 4c_1 + 16c_2
\]

\[
b_3 = \frac{a_3}{a_2} = 12 \Rightarrow 12 = c_1 2^3 + c_2 4^3 = 8c_1 + 64c_2.
\]

Thus, \(c_1 = -\frac{1}{2}, c_2 = \frac{1}{4}\) and

\[
b_n = -2^{n-1} + 2^{n-2} = 2^{n-1} (2^{n-1} - 1)
\]

\[
a_n = b_n a_{n-1} = b_n b_{n-1} a_{n-2} = \cdots
\]

\[
= b_n b_{n-1} \cdots b_2 a_1 = b_n b_{n-1} \cdots b_2
\]

\[
= 2^{(n-1)+(n-2)+\cdots+1} \prod_{i=1}^{n-1} (2^i - 1)
\]

\[
= 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1).
\]

We need to show that \(n\) divides \(a_n = 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1)\). Write \(n = 2^j k\) where \(k\) is odd. Certainly \(j \leq n \leq n(n-1)/2\) for \(n \geq 2\) so that \(2^j\) divides \(2^{n(n-1)/2}\). It suffices to show that
We recall that since \( \gcd(k, 2) = 1 \), \( k \) divides \( 2^{\phi(k)} - 1 \) and \( \phi(k) \) is less than \( n \), as required.

**B1.** Right triangle \( ABC \) has right angle at \( C \) and \( \angle BAC = \theta \); the point \( D \) is chosen on \( AB \) so that \( |AC| = |AD| = 1 \); the point \( E \) is chosen on \( BC \) so that \( \angle CDE = \theta \). The perpendicular to \( BC \) at \( E \) meets \( AB \) at \( F \). Evaluate \( \lim_{\theta \to 0} |EF| \).

**Solution.** Note that \( |AB| \cos \theta = |AC| = 1 \), so that

\[
b := |AB| = 1 / \cos \theta = \sec \theta.
\]

The line \( CB \) has slope

\[
\frac{-\sin \theta}{b - \cos \theta} = \frac{-\sin \theta}{1 - \cos^2 \theta} = \frac{-\sin \theta \cos \theta}{\sin^2 \theta} = -\cot \theta
\]

and hence its equation is

\[
y = -\cot \theta (x - b) = \frac{-x \cos \theta + 1}{\sin \theta}.
\]

Also \( \angle ADC = (\pi - \theta) / 2 \) and

\[
\angle EDB = \pi - \theta - \angle ADC = \pi - \theta - (\pi - \theta) / 2 = (\pi - \theta) / 2.
\]

Thus, the line \( DE \) has equation

\[
y = \tan \left( \frac{\pi}{2} - \frac{\theta}{2} \right) = \cot \left( \frac{\theta}{2} \right) (x - 1) = \frac{\sin \theta}{1 - \cos \theta} (x - 1)
\]

The lines intersect when

\[
\frac{\sin \theta}{1 - \cos \theta} (x - 1) = \frac{-x \cos \theta + 1}{\sin \theta}
\]

\[
(x - 1) \sin^2 \theta = (-x \cos \theta + 1) (1 - \cos \theta)
\]

\[
(x - 1) (1 + \cos \theta) = -x \cos \theta + 1
\]

\[
x (1 + \cos \theta) - (1 + \cos \theta) = -x \cos \theta + 1
\]

\[
x (1 + 2 \cos \theta) = 2 + \cos \theta
\]

\[
x = \frac{2 + \cos \theta}{1 + 2 \cos \theta}
\]

Then

\[
y = -\left( \cot \theta \right) \frac{2 + \cos \theta}{1 + 2 \cos \theta} + \frac{1}{\sin \theta}.
\]
Finally,

\[ |EF| = \frac{y}{\sin \theta} = -\frac{\cot \theta \cdot 2 + \cos \theta}{\sin \theta \cdot 1 + 2 \cos \theta} + \frac{1}{\sin^2 \theta} \]

\[ = -\frac{\cos \theta \cdot \frac{2 + \cos \theta}{1 + 2 \cos \theta} + 1}{\sin^2 \theta} \]

\[ = -\frac{(\cos \theta) \cdot (2 + \cos \theta) + 1 + 2 \cos \theta}{(\sin^2 \theta) \cdot (2 + \cos \theta)} \]

\[ = \frac{1}{2 + \cos \theta} \rightarrow \frac{1}{3} \text{ as } \theta \rightarrow 0. \]

**B2.** Let \( P(x) \) be a polynomial of degree \( n \) such that \( P(x) = Q(x)P''(x) \), where \( Q(x) \) is a quadratic polynomial and \( P''(x) \) is the second derivative of \( P(x) \). Show that if \( P(x) \) has at least two distinct roots then it must have \( n \) distinct roots.

**Solution.** Suppose that \( P(x) \) does not have \( n \) distinct roots. Then \( P(x) \) is of the form

\[ P(x) = (x - a)^m R(x) \]

for some \( m \geq 2 \) where \( R(a) \neq 0 \) and \( a \in \mathbb{C} \). Then

\[ P''(x) = (m \cdot (x - a)^{m-1} R(x) + (x - a)^m R'(x))' \]

\[ = m \cdot (m - 1) \cdot (x - a)^{m-2} R(x) + 2m \cdot (x - a)^{m-1} R'(x) + (x - a)^m R''(x) \]

\[ = (x - a)^{m-2} \left( m \cdot (m - 1) \cdot R(x) + 2m \cdot (x - a) \cdot R'(x) + (x - a)^2 \cdot R''(x) \right) \]

\[ = (x - a)^{m-2} \cdot S(x) \]

Since \( S(a) = m \cdot (m - 1) \cdot R(a) \neq 0 \), \( S(x) \) has no factors of \( (x - a) \). Then

\[ (x - a)^m R(x) = P(x) = Q(x)P''(x) = Q(x) \cdot (x - a)^{m-2} \cdot S(x) \]

implies \( Q(x) = c \cdot (x - a)^2 \) and

\[ R(x) = cS(x) \]

Then

\[ R(a) = cS(a) = cm \cdot (m - 1) \cdot R(a) \Rightarrow c = \frac{1}{m \cdot (m - 1)}. \]

Thus,

\[ P(x) = Q(x)P''(x) = \frac{1}{m \cdot (m - 1)} \cdot (x - a)^2 \cdot P''(x) \]

or

\[ P''(x) = m \cdot (m - 1) \cdot \frac{P(x)}{(x - a)^2} \]

However, by expanding \( P(x) \) in powers of \( (x - a) \) we can see that this is the case only if \( P(x) = b \cdot (x - a)^m \) for some constant \( b \).
B3. Let \( A = \{(x, y) : 0 \leq x, y < 1\} \). For \((x, y) \in A\), let

\[ S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n, \]

where the sum ranges over all pairs \((m, n)\) of positive integers satisfying the indicated inequalities. Evaluate

\[ \lim_{(x, y) \to (1, 1), (x, y) \in A} (1 - xy^2)(1 - x^2 y)S(x, y). \]

Solution. Note that

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^m y^n = \sum_{m=1}^{\infty} x^m \sum_{n=1}^{\infty} y^n = \frac{x}{1 - x} \frac{y}{1 - y} \]

Moreover,

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^m y^n - S(x, y) \]
\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^m y^n - \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n \]
\[ = \sum_{m \geq 2n+1} \sum_{n \geq m+1} x^m y^n + \sum_{m \geq 2n+1} \sum_{n \geq m+1} x^m y^n \]
\[ = \sum_{n=1}^{\infty} \frac{x^{2n+1}}{1 - x} y^n + \sum_{m=1}^{\infty} \frac{y^{2m+1}}{1 - y} x^m \]
\[ = \frac{x}{1 - x} \sum_{n=1}^{\infty} (x^2 y)^n + \frac{y}{1 - y} \sum_{m=1}^{\infty} (y^2 x)^m \]
\[ = \frac{x^3 y}{(1 - x)(1 - x^2 y)} + \frac{y^3 x}{(1 - y)(1 - y^2 x)} \]

Thus,

\[ S(x, y) = \frac{x}{1 - x} \frac{y}{1 - y} - \frac{x^3 y}{(1 - x)(1 - x^2 y)} - \frac{y^3 x}{(1 - y)(1 - y^2 x)} \]
\[ = xy ((1 - x^2 y)(1 - y^2 x) - x^3 (1 - y^2 x)(1 - y) - y^2 (1 - x^2 y)(1 - x)) \]
\[ = xy \left( \frac{1 - x^3 y^3 - x^2 + x^3 y^2 - y^2 + x^2 y^3}{(1 - x)(1 - y)(1 - x^2 y)(1 - y^2 x)} \right) \]
\[ = xy \left( \frac{1 - x (y + x + 1 - x^2 y^2)}{(1 - x)(1 - y)(1 - x^2 y)(1 - y^2 x)} \right) \]
\[ = \frac{xy y + x + 1 - x^2 y^2}{(1 - x^2 y)(1 - y^2 x)} \]
and
\[
\lim_{(x,y)\to(1,1),(x,y)\in A} (1 - xy^2)(1 - x^2y)S(x, y) = 3.
\]

**B4.** Let \( f \) be a real function with a continuous third derivative such that \( f(x), f'(x), f''(x), f'''(x) \) are positive for all \( x \). Suppose that \( f'''(x) \leq f(x) \) for all \( x \). Show that \( f'(x) < 2f(x) \) for all \( x \).

**Solution.** Let \( c = \lim_{x \to -\infty} f(x) \geq 0 \). By replacing \( f(x) \) by \( f(x) - c \) we may assume \( c = 0 \). Let \( c' = \lim_{x \to -\infty} f'(x) \geq 0 \). Then \( f'(x) > d > 0 \) and for \( x < 0 \)

\[
f(0) = f(x) + \int_{x}^{0} f'(x)d\xi > f(x) + c'x.
\]

Thus,
\[
f(x) < f(0) - c'x
\]
and it follows that \( c' = 0 \). Similarly, \( c'' = \lim_{x \to -\infty} f''(x) = 0 \). Since \( f'''(x) \leq f(x) \),

\[
f''(x)f'''(x) \leq f''(x)f(x) < f''(x)f(x) + f'(x)^2
\]

or \( 0 < \frac{d}{dx} \left( f'(x)f(x) - \frac{1}{2}f''(x)^2 \right) \).

Also, as

\[
\lim_{x \to -\infty} \left( f'(x)f(x) - \frac{1}{2}f''(x)^2 \right) = c'c - \frac{1}{2}(c'')^2 = 0,
\]

we then have

\[
f'(x)f(x) > \frac{1}{2}f''(x)^2.
\]

Thus,

\[
\frac{d}{dx} \left( \frac{3}{4}f(x)^2 \right) = \frac{3}{2}f(x)f'(x) > f'(x)f(x) + \frac{1}{2}f'(x)f''(x)
\]

\[
> \frac{1}{2} \left( f''(x)^2 + f'(x)f''(x) \right)
\]

\[
= \frac{d}{dx} \left( \frac{1}{2}f'(x)f''(x) \right)
\]

or

\[
\frac{d}{dx} \left( \frac{3}{4}f(x)^2 - \frac{1}{2}f'(x)f''(x) \right) > 0.
\]

Since \( \lim_{x \to -\infty} \left( \frac{3}{4}f(x)^2 - \frac{1}{2}f'(x)f''(x) \right) = \frac{3}{4}c^2 - \frac{1}{2}c'c'' = 0 \), we then have

\[
\frac{3}{4}f(x)^2 - \frac{1}{2}f'(x)f''(x) > 0.
\]

Hence,

\[
\frac{3}{4}f(x)^2 > \frac{1}{2}f''(x)^2
\]

or

\[
\frac{1}{4} \frac{d}{dx} \left( f(x)^3 \right) > \frac{1}{6} \frac{d}{dx} \left( f'(x)^3 \right).
\]

Thus,

\[
f(x)^3 > \frac{2}{3}f'(x)^3
\]
and
\[ f'(x) < \left( \frac{3}{2} \right)^{1/3} f(x) < 2f(x). \]

**B5.** For an integer \( n \geq 3 \), let \( \theta = 2\pi/n \). Evaluate the determinant of the \( n \times n \) matrix \( I + A \), where \( I \) is the \( n \times n \) identity matrix and \( A = (a_{jk}) \) has entries \( a_{jk} = \cos(j\theta + k\theta) \) for all \( j, k \).

**Solution.** Let \( z = e^{i\theta} \). Then
\[
\cos(m\theta) = \frac{1}{2} (e^{im\theta} + e^{-im\theta}) = \frac{1}{2} (z^m + z^{-m}) = \frac{1}{2} (z^m + \overline{z}^{-m}).
\]
Let \( v = (z, z^2, z^3, \ldots, z^n) \) and \( \overline{v} = (\overline{z}, \overline{z}^2, \overline{z}^3, \ldots, \overline{z}^n) \). The matrix \( A \) is half the sum of the outer products \( v \otimes v \) and \( \overline{v} \otimes \overline{v} \). Thus,
\[
I + A = I + \frac{1}{2} (v \otimes v + \overline{v} \otimes \overline{v})
\]
For any \( w \in \mathbb{C}^n \),
\[
(I + A) w = w + \frac{1}{2} ((v \cdot w) v + (\overline{v} \cdot w) \overline{v})
\]
where \( v \cdot w \) is the usual bilinear dot product. For \( u \in \mathbb{C}^n \) orthogonal to both \( v \) and \( \overline{v} \), we have
\[
(I + A) u = u.
\]
This shows that \( I + A \) is the identity on \( \text{span} (v, \overline{v}) \). Thus, we need only evaluate \( \det(I + A) \) on \( \text{span} (v, \overline{v}) \). Note that
\[
(I + A) (v) = v + \frac{1}{2} ((v \cdot v) v + (\overline{v} \cdot v) \overline{v}) = (1 + \frac{1}{2} (v \cdot v)) v + \frac{1}{2} (\overline{v} \cdot v) \overline{v}
\]
\[
(I + A) (\overline{v}) = \overline{v} + \frac{1}{2} ((v \cdot \overline{v}) v + (\overline{v} \cdot \overline{v}) \overline{v}) = \frac{1}{2} (v \cdot \overline{v}) v + (1 + \frac{1}{2} (v \cdot \overline{v})) \overline{v}
\]
Thus, the matrix of \( I + A \) on \( \text{span} (v, \overline{v}) \) is
\[
\begin{bmatrix}
1 + \frac{1}{2} (v \cdot v) & \frac{1}{2} (v \cdot \overline{v}) \\
\frac{1}{2} (v \cdot \overline{v}) & 1 + \frac{1}{2} (\overline{v} \cdot \overline{v})
\end{bmatrix}
\]
and the determinant is (since \( z^{2n} = 1 \) and \( z^{2n} = 1 \))
\[
\begin{vmatrix}
1 + \frac{1}{2} (v \cdot v) & \frac{1}{2} n \\
\frac{1}{2} n & 1 + \frac{1}{2} (\overline{v} \cdot \overline{v})
\end{vmatrix}
= \begin{vmatrix}
1 + \frac{1}{2} (z^2 + \cdots + z^{2n}) & \frac{1}{2} n \\
\frac{1}{2} n & 1 + \frac{1}{2} (\overline{z}^2 + \cdots + \overline{z}^{2n})
\end{vmatrix}
= \begin{vmatrix}
1 + \frac{1}{2} \left( z^{2^{1-2n}} \right) & \frac{1}{2} n \\
\frac{1}{2} n & 1 + \frac{1}{2} \left( \overline{z}^{2^{1-2n}} \right)
\end{vmatrix}
= \begin{vmatrix}
\frac{1}{2} n & 1 \\
\frac{1}{2} n & 1
\end{vmatrix}
= 1 - \frac{n^2}{4}.\]
Let $S$ be a finite set of integers, each greater than 1. Suppose that for each integer $n$ there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

**Solution.** Let $p_1 < p_2 < \ldots < p_k$ be the set of all of the primes that occur in at least one of the prime factorizations of the elements of $S$. $S_1 \subseteq S$ be the subset consisting of all those members of $S$ involving $p_1$ in their prime factorization. If there are $s, t \in S_1$ with $\gcd(s, t) = p_1$, then we are done. If not, let $p_{i_1} = p_1$ and let $p_{i_2} > p_1$ be the smallest prime that occurs in the factorization of members of $S - S_1$, and let $S_2 \subseteq S - S_1$ be the subset consisting of all those members of $S$ involving $p_{i_2}$ in their prime factorization. If there are $s, t \in S_2$ with $\gcd(s, t) = p_{i_2}$, then we are done. We continue in this way obtaining $p_{i_1} < p_{i_2} < \ldots < p_{i_j}$ where the process stops at some $j \leq k$. Note that we either have $s, t \in S_j$ with $\gcd(s, t) = p_{i_j}$ (in which case we are done) or $S = \bigcup_{m=1}^{j} S_m$. Suppose that $S = \bigcup_{m=1}^{j} S_m$. Let $n = p_{i_1}p_{i_2}\cdots p_{i_j}$. Then by assumption, there is some $s \in S$, such that

$$\gcd(s, n) = s \text{ or } \gcd(s, n) = 1.$$ 

Now $\gcd(s, n) = 1$ is impossible, since $S = \bigcup_{m=1}^{j} S_m$. Thus, $\gcd(s, n) = s$, and so $s$ is a divisor of $n = p_{i_1}p_{i_2}\cdots p_{i_j}$. Let $j'$ be the smallest integer so that there is a divisor $s \in S$ of $p_{i_1}p_{i_2}\cdots p_{i_{j'}}$. Now $p_{i_{j'}}$ must occur (exactly once) as a factor of $s$ or else $j'$ can be reduced. Select $s_{j'} \in S_{j'}$. We claim

$$\gcd(s, s_{j'}) = p_{i_{j'}}.$$ 

Indeed, by definition $s_{j'}$ has factor $p_{i_{j'}}$, but no factors of $p_{i_1}, p_{i_2}, \ldots p_{i_{j'-1}}$. 

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