University of Hawai'i at Mānoa — Department of Mathematics
W. Erb — Spring 2018

Math 253A - Accelerated Calculus III

Homework sheet 4 **Due 02/09/2018**

To read: Section 12.3, 12.4, 12.5 and 12.6 in the book.

Problem 1 (Length is independent of parametrization)
To illustrate that the length of a smooth curve in space does not depend on the used parametrization, calculate the length of one turn of the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $t \in [0, 2\pi]$, using the following parametrizations. Show the explicit calculation for all four length integrals.

a) $\vec{r}(t) = \langle \cos t, -\sin t, -t \rangle$, $t \in [-2\pi, 0]$.

b) $\vec{r}(t) = \langle \cos t^2, \sin t^2, t^2 \rangle$, $t \in [0, \sqrt{2\pi}]$.

c) $\vec{r}(t) = \langle \cos 4t, \sin 4t, 4t \rangle$, $t \in [0, \pi/2]$.

d) $\vec{r}(t) = \langle \cos(e^t), \sin(e^t), e^t \rangle$, $t \in (-\infty, \ln(2\pi)]$.

Problem 2
Consider the helix given by

$$\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle, \quad t \geq 0,$$

with some constants $a, b > 0$.

a) Calculate the unit vectors $\vec{T}(t)$, $\vec{N}(t)$ and the curvature $\kappa(t)$.

b) Show that the torsion of the helix is given by $\tau = b/(a^2 + b^2)$. For this calculate the binormal $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ and its derivative $\vec{B}'(t)$. Then, the torsion is given by the formula

$$\tau(t) = -\frac{\vec{B}'(t) \cdot \vec{N}(t)}{|\vec{r}\,'(t)|}.$$

What is the largest value τ can have for a fixed value of a? Give reasons for your answer.

Problem 3

a) For the curve

$$\vec{r}(t) = (\cos t + t \sin t)\vec{i} + (\sin t - t \cos t)\vec{j},$$

calculate the arc length parameter $s(t)$ starting from the initial location $\vec{r}(0) = \langle 1, 0, 0 \rangle$, i.e. calculate the integral $s(t) = \int_0^t |\vec{r}\,'(\tau)| d\tau$. What is the length of the curve for the portion $t \in [\pi/2, \pi]$?

b) Consider the curve

$$\vec{r}(t) = \langle t \cos t, t \sin t, t^2 \rangle.$$

For $t = 0$, give the acceleration $\vec{a}(0) = \vec{r}\,''(0)$ in the form $a_T \vec{T}(0) + a_N \vec{N}(0)$.

Problem 4
Read the section on Kepler's Third law in the book (page 689). Then, find the length of the major axis a of Earth's orbit using Kepler's third law and the fact that Earth's orbital period is 365.256 days.