(1) Recall the correspondence between vector fields and differential forms on an open subset $B \subset \mathbb{R}^3$, and the diagram from class relating the exterior derivative $d$ to the familiar grad, div and curl from calculus. Prove Theorem 31.2 on p.264 of Munkres. We did some of this in class.

(2) Exercise 2, p.260 of Munkres.

(3) While we did not discuss this in class, a differential form $\omega \in \Omega^k(B)$ is closed if $d\omega = 0$, and is exact if there is a differential form $\sigma \in \Omega^{k-1}(B)$ such that $\omega = d\sigma$. Verify the following assertions, concerning the differential form $\omega$ on various open subsets of $\mathbb{R}^2$:

\[ \omega = \frac{-y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy. \]

a) $\omega \in \Omega^1(B)$ is closed, $B = \mathbb{R}^2 \setminus \{ (0, 0) \}$

b) $\omega \in \Omega^1(B)$ is not exact.

c) $\omega \in \Omega^1(A)$ is exact, $A = \{ (x, y) : x > 0 \} \subset \mathbb{R}^2$.

For (c) exhibit a smooth function $\varphi$ on $A$ such that $d\varphi = \omega$. For (b) you have to show there is no such smooth function on $B$. Hint: Stokes’ theorem for line integrals.

(4) (The Poincaré Lemma) Let $\mathbb{D}$ be the open unit disk in the $(x, y)$-plane. Define a smooth 2-form $\eta \in \Omega^2(\mathbb{D})$ by

\[ \eta = \frac{1}{\sqrt{1 - x^2 - y^2}} \, dx \wedge dy \]

The disk $\mathbb{D}$ is an example of a ‘star convex region’ (see p.330-332 in Munkres). According to the Poincaré Lemma (Theorem 39.3 in Munkres), we must necessarily have that $\eta = d\sigma$ for some smooth 1-form $\sigma \in \Omega^1(\mathbb{D})$. This problem outlines an approach to finding such a form.

Write $f(x, y) = (1 - x^2 - y^2)^{-1/2}$ so that $\eta = f \, dx \wedge dy$. Define

\[ \sigma(x, y) = \left( \int_0^1 t f(tx, ty) \, dt \right) \, (xdy - ydx); \]

the integral here is an ordinary calculus integral. It happens that $\sigma \in \Omega^1(\mathbb{D})$. Calculate $\sigma$ explicitly and verify that $d\sigma = \eta$. 