MATH 411 – HOMEWORK

INSTRUCTIONS
Write your solutions on loose leaf paper. Please at most one problem per page. Staple this problem sheet and your solution sheets together. Return on Thursday, October 9 during class. If you are unable to attend class put your homework in my mailbox.

You will be graded not only on the correctness of your solutions, but also on the quality of the exposition and organization of your proofs.

(15) 1. Define operations of addition and scalar multiplication (by elements of $\mathbb{R}$) on the set $V = \mathbb{R}^+ = \{x \in \mathbb{R}: x > 0\}$ by

   (1) $x +_V y = xy$, for $x, y \in V$
   (2) $a \cdot_V x = x^a$, for $x \in V$ and $a \in \mathbb{R}$

   (5) a) Verify that with these operations $V$ is a vector space over $\mathbb{R}$.
   (5) b) Is the set of positive rational numbers $\mathbb{Q}^+ \subset V$ a subspace?

(17) 2. Let $V$ be a vector space and let $P : V \to V$ be a linear transformation with the property that $P^2 = P$.

   (7) a) Show that $V = \text{Null } P \oplus \text{Range } P$.

An $n \times n$ matrix $A$ is symmetric if $A^t = A$; it is skew-symmetric if $A^t = -A$. Let Sym$_n(\mathbb{R})$ and Skew$_n(\mathbb{R})$ denote the vector spaces of symmetric and skew-symmetric $n \times n$ matrices, respectively.

   (4) b) Show that Sym$_n(\mathbb{R})$ and Skew$_n(\mathbb{R})$ are subspaces of Mat$(n, n, \mathbb{R})$.
   (6) c) Define a linear transformation $P : \text{Mat}(n, n, \mathbb{R}) \to \text{Mat}(n, n, \mathbb{R})$ with the following properties
   
   $P^2 = P$,  \hspace{1cm} \text{Range } P = \text{Sym}_n(\mathbb{R})$,  \hspace{1cm} \text{and } \text{Null } P = \text{Skew}_n(\mathbb{R})$.

   What can you conclude?

(13) 3. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Define a function $\delta_A : \text{Mat}(2, 2, \mathbb{R}) \to \text{Mat}(2, 2, \mathbb{R})$ by $\delta_A(X) = AX - XA$, for $X \in M_2(\mathbb{R})$.

   (4) a) Show that $\delta_A$ is a linear transformation
   (5) b) Describe the null space of $\delta_A$.
   (4) c) Compute dim Null($\delta_A$) and dim Range($\delta_A$).
4. A linear transformation $T \in \mathcal{L}(V)$ is nilpotent if there exists $m > 0$ such that $T^m = 0$. Prove the following statements:

(5) a) If $T$ is nilpotent then $1 - T$ is invertible. 
*Hint: write down an inverse.*

(5) b) If there exists a basis $\alpha = (v_1, \ldots, v_n)$ of $V$ such that $v_1 \in \operatorname{Null}(T)$ and $T(v_k) \in \operatorname{span}(v_1, \ldots, v_{k-1})$ for all $1 < k \leq n$ then $T$ is nilpotent.

Let $V = P_n(\mathbb{C})$ and let $T(f) = f' \ (\text{the derivative})$; we saw $T \in \mathcal{L}(V)$.

(5) b) Let $\beta = (p_0, \ldots, p_n)$ be the basis defined by $p_k(x) = 1 + \ldots + x^k$. Compute the matrix of $T$ with respect to $\beta$.

5. Let $U$ and $W$ be dimensional subspaces of $\mathbb{R}^n$. We proved in class that 
\[ \dim(U + W) + \dim(U \cap W) = \dim U + \dim W. \]

Give a proof of this fact using the Rank-Nullity Theorem by defining a linear transformation $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $m = \dim U + \dim W$, such that

(1) $\operatorname{Range}(T) = U + W$
(2) $\operatorname{Null}(T) \cong U \cap W$

Explain how the equality follows.
*Hint: $T$ will be multiplication by an $n \times m$ matrix built from bases of $U$ and $W$.**