Partial Orders: Let $\leq$ be a partial order on a set $S$ and let $T \subset S$.

a) Define what we mean by an upper bound for $T$.
b) Define what we mean by a least upper bound for $T$.
c) Define what we mean by an order isomorphism.
d) Let $A$ be a set, let $S = \mathcal{P}(A)$ partially ordered by inclusion, and let $T \subset S$. Prove that $\bigcup_{x \in T} x$ is the least upper bound for $T$.
e) Prove that $\mathcal{P}\{1, 2\}$, ordered by inclusion, is order isomorphic to $\{\text{factors of 6}\}$, ordered by the relation “divides”.

Total Orders: Let $\leq$ be a total order on a set $S$.

a) Define what we mean by a total order.
b) Does there exist an injective order preserving map $\mathcal{P}\{1, 2\} \rightarrow \mathbb{N}$? Explain.
c) Explain why no two of the sets $\{1, 2\}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are order isomorphic. Each set is totally ordered by the usual $\leq$.

Functions: Let $f : A \rightarrow B$ and $g : B \rightarrow C$. For each statement, give a proof or a counterexample.

a) If $f$ and $g$ are injective then $g \circ f$ is injective.
b) If $g \circ f$ is injective then $f$ is injective.
c) If $S, T \subset A$ then $f(S \cup T) = f(S) \cup f(T)$.
d) If $S, T \subset B$ then $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
e) If $S \subset A$ then $f^{-1}(f(A)) = A$.

Cardinality: Let $A$, $B$ and $C$ be sets.

a) Define what we mean by $\text{card}(A) = \text{card}(B)$.
b) Define what we mean by $\text{card}(A) \leq \text{card}(B)$ and by $\text{card}(A) < \text{card}(B)$.
c) State the Schroeder-Bernstein Theorem.
d) Proof or counterexample: if $A \subset B$ then $\text{card}(A) < \text{card}(B)$?
e) Proof that if $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(C)$ then $\text{card}(A) \leq \text{card}(C)$.
f) Give an example of a finite set, a countably infinite set and an uncountable set.
PREPARE SOLUTION TO THE FOLLOWING PROBLEMS.
ONE WILL APPEAR ON THE FINAL EXACTLY AS IT APPEARS BELOW.

**Problem:** Give an example of a finite graph, vertices \( v \) and \( w \) and two different paths from \( v \) to \( w \) that pass through the same vertices (necessarily in a different order).

Show that if a graph contains two vertices \( v \) and \( w \) and two different paths connecting \( v \) to \( w \) then the graph contains a cycle. *Hint: the proof outlined in class had a tiny flaw; compare to the above example.*

**Problem:** Show that if the sets \( \{1, 2, \ldots, m\} \) and \( \{1, 2, \ldots, n\} \) are in bijective correspondence then \( n = m \). *Hint: induct on \( n \).*

**Problem:** Show that a finite totally ordered set is order isomorphic to \( \{1, 2, \ldots, n\} \) with the usual order, for some \( n \). Exhibit an infinite totally ordered set \( S \) that is in bijective correspondence with \( \mathbb{N} \) but is not order isomorphic to \( \mathbb{N} \) with the usual order.