HOMEWORK: OPERATORS

Unbounded functionals: Let $H$ be a separable infinite dimensional Hilbert space. ‘Construct’ an unbounded linear functional $H \to \mathbb{C}$. Hint: an honest vector space basis of $H$ is uncountable.

Infinite matrices: Show that the infinite matrix
\[
\begin{pmatrix}
t_1 & 0 & \ldots \\
0 & t_2 & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]
defines, by ordinary matrix multiplication, a bounded linear operator on $\ell^2(\mathbb{N}) \iff \sup |t_i| < \infty$; in which case this supremum is the norm of the operator. More generally, suppose the infinite matrix
\[
\begin{pmatrix}
t_{11} & t_{12} & \ldots \\
t_{21} & t_{22} & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]
satisfies the following condition: there exists constants $\alpha$ and $\beta$ such that
\[(a) \sum_i |t_{ij}| \leq \alpha \text{ for every } j
(b) \sum_j |t_{ij}| \leq \beta \text{ for every } i.
\]
Then $(t_{ij})$ defines, by ordinary matrix multiplication, a bounded operator on $\ell^2(\mathbb{N})$ of norm not greater than $\sqrt{\alpha \beta}$. This is called the Schur test.

Operators and sesquilinear forms: Let $H$ be a Hilbert space. A sesquilinear form on $H$ is a function

$$(\cdot, \cdot) : H \times H \to \mathbb{C}$$

that is linear in the first variable and conjugate linear in the second variable. A sesquilinear form is bounded if there exists a constant $C$ such that

$$|(x, y)| \leq C \|x\|\|y\|,$$

for all $x, y \in H$; in this case, the norm of the functional is the infimum over all $C$ for which the inequality is satisfied. Prove:

(a) if $T$ is a bounded operator then $(x, y) = \langle T(x), y \rangle$ is a bounded sesquilinear form; its norm is the operator norm of $T$

(b) every bounded sesquilinear form arises from a bounded operator via the construction in (a)

Sesquilinear forms behave like inner products.

(c) formulate and prove a polarization rule for sesquilinear forms

(d) prove that if $(\cdot, \cdot)$ and $[\cdot, \cdot]$ are sesquilinear forms and if $(x, x) = [x, x]$ for all $x \in H$ then $(x, y) = [x, y]$ for all $x, y \in H$

A sesquilinear form is positive if $(x, x) \geq 0$ for all $x \in H$.

(e) formulate and prove a Cauchy-Schwarz inequality for positive sesquilinear forms
**Strong operator topology:** Let $H$ be a Hilbert space, and let $\mathcal{B}(H)$ be the collection of bounded operators on $H$. Let

$$U(T; x_1, \ldots, x_n; \varepsilon) = \{ S \in \mathcal{B}(H) : \|Tx_i - Sx_i\| < \varepsilon, \text{ for } i = 1, 2, \ldots, n \}.$$ 

Verify the following:

(a) the collection of $U(T; x_1, \ldots, x_n; \varepsilon)$ as $T$, $x_i$ and $\varepsilon$ vary, are the basis of a topology on $\mathcal{B}(H)$

(b) the collection of $U(T; x_1, \ldots, x_n; \varepsilon)$ as $x_i$ and $\varepsilon$ vary, are a neighborhood basis at $T \in \mathcal{B}(H)$

(c) a net $T_\lambda$ converges to $T$ in this topology $\iff T_\lambda(x)$ converges to $T(x)$ for every $x \in H$

This topology is the **strong operator topology**. NB: this problem is not stated very well – be careful with ‘basis’ and ‘basis at $T$', etc.

The strong operator topology is metrizable when restricted to bounded subsets of $\mathcal{B}(H)$ if and only if $H$ is separable. The strong operator topology is *not* metrizable on all of $\mathcal{B}(H)$, even when $H$ is separable.

**The unitary group:** Let $H$ be a Hilbert space. Show that the collection of unitary operators on $H$, denoted $\mathcal{U}(H)$, form a group. Show that the group operations

$$U \mapsto U^{-1} : \mathcal{U}(H) \to \mathcal{U}(H)$$

$$(U, V) \mapsto UV : \mathcal{U}(H) \times \mathcal{U}(H) \to \mathcal{U}(H)$$

are continuous when $\mathcal{U}(H)$ is given the strong operator topology. Succinctly stated, $\mathcal{U}(H)$ is a **topological group** when given the strong operator topology.

Prove that the adjoint is *not* continuous for the strong operator topology as a map $\mathcal{B}(H) \to \mathcal{B}(H)$. Nevertheless, it *is* continuous for the strong operator topology as a map $\mathcal{U}(H) \to \mathcal{U}(H)$, as you are asked to prove above.

**Invertibility:** A bounded operator $T$ on $H$ is **invertible** if there exists a bounded operator $S$ on $H$ such that $ST = 1 = TS$; the $S$ satisfying this condition is unique, and is denoted $T^{-1}$. Prove that $T$ is invertible $\iff$ both of the following hold:

(a) $T$ is bounded below: $\exists C > 0$ such that $\forall x \in H$ we have $\|T(x)\| \geq C\|x\|$

(b) $T$ is densely onto: $\text{ran}(T) = H$

**Hint:** By the Bounded Inverse Theorem (or Closed Graph Theorem if you prefer) an operator is invertible $\iff$ it is injective and surjective. For each of the following either give an example or prove there is no such example:

(c) $\ker(T) = 0$ but $T$ is not bounded below

(d) $T$ is bounded below but $\ker(T) \neq 0$

(e) $T$ is bounded below, but not densely onto

(f) $T$ is bounded below, densely onto but not onto

Thinking about the examples may help with the first part of the problem.