

A SYMPLECTIC ANALOGUE OF THE MOSTOW-PALAIS THEOREM

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ABSTRACT. We show that given a Hamiltonian action of a compact and connected Lie group G on a symplectic manifold (M, ω) of finite type, there exists a linear symplectic action of G on some \mathbf{R}^{2n} equipped with its standard symplectic structure such that (M, ω, G) can be realized as a reduction of this \mathbf{R}^{2n} with the induced action of G .

Introduction.

Whitney's embedding theorem tells us that every manifold M can be realized as a closed submanifold of some \mathbf{R}^n . The equivariant version of this result is the Mostow-Palais theorem ([Mo],[Pa]), which states that if a compact Lie group G acts smoothly on M with finitely many orbit types, then M can be equivariantly embedded in some orthogonal representation space for G . That is, there exists an \mathbf{R}^n with an orthogonal action of G such that M is realized as a closed G -invariant submanifold of this \mathbf{R}^n and the induced action of G on M coincides with the original action. One can ask to what extent analogous results hold in symplectic geometry. But in the symplectic category there is no general embedding theorem; only when one restricts to special classes of symplectic manifolds is it possible to obtain a strict analogy with the differentiable category. For instance, a result of Gromov and Tischler ([Gr],[Ti]) asserts that every integral compact symplectic manifold can be symplectically embedded in some \mathbf{CP}^n with its usual Kähler structure. Ono has proven an equivariant version of this theorem under the assumption that the compact group action can be lifted to some prequantization of (M, ω) [On]. Similar results for open symplectic manifolds can be found in [Gr].

However, in the symplectic category the most interesting construction is not embedding, but *reduction*. We say that a symplectic manifold $(\bar{M}, \bar{\omega})$ is obtained as the reduction of a symplectic manifold (M, ω) by the submanifold $C \subset M$ if the pullback ω_C of ω to

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C has constant rank, its characteristic distribution $\ker \omega_C$ is fibrating and the quotient (symplectic) manifold $C/\ker \omega_C$ is symplectomorphic to $(\overline{M}, \overline{\omega})$. In this context one has the following symplectic analogue of the Whitney embedding theorem.

Theorem 0 ([GT]). *If M is a symplectic manifold of finite type, then M can be realized as a reduction of some \mathbf{R}^{2n} with its standard symplectic structure.*

Now if $(\overline{M}, \overline{\omega})$ is obtained as the reduction of (M, ω) by C , and if C is invariant under the symplectic action of a Lie group G on M , then we obtain an induced symplectic action of G on $(\overline{M}, \overline{\omega})$. We then say that the triple $(\overline{M}, \overline{\omega}, G)$ is obtained as the *equivariant reduction* of (M, ω, G) by C . With these preparations we now can state the main result of this paper.

Theorem 1. *Let there be given a symplectic action of a compact and connected Lie group G on (M, ω) , where M is of finite type. If the G -action admits a momentum map, then (M, ω, G) can be obtained as an equivariant reduction of some \mathbf{R}^{2n} with its standard symplectic structure. Moreover, one may suppose that the G -action on \mathbf{R}^{2n} is the cotangent lift of an orthogonal action of G on \mathbf{R}^n .*

Suppose $(\overline{M}, \overline{\omega}, G)$ is obtained as an equivariant reduction of (M, ω, G) . Then one easily shows that if there exists a momentum map for the action on M , then there exists an induced one for the action on \overline{M} . Moreover, if the momentum map on M is Ad^* -equivariant, then so is the one on \overline{M} . Thus the natural symplectic analogue of the Mostow-Palais theorem is not for *symplectic* G -actions, but rather for *Hamiltonian* G -actions, because a cotangent G -action on \mathbf{R}^{2n} always admits an Ad^* -equivariant momentum map. Actually, in view of the following result, we see that the natural symplectic analogue of the Mostow-Palais theorem is for *weakly hamiltonian* actions.

Proposition 2. *Suppose that a compact Lie group G acts smoothly on a manifold M , leaving a closed 2-form ω invariant. If there exists a momentum map for this action, then there exists an Ad^* -equivariant momentum map.*

Proof of Theorem 1.

Theorem 0 was proved in [GT], and we briefly recall the three steps of that proof because we need some terminology and since the proof of Theorem 1 proceeds along the same lines. In step 1 one realizes (M, ω) as a reduction of T^*M with the modified symplectic structure $\omega_M = d\theta_M + \tau_M^* \omega$, where $\tau_M : T^*M \rightarrow M$ is the canonical projection and θ_M the canonical 1-form on T^*M . We call any cotangent bundle T^*M with such a symplectic form (where ω is a closed 2-form on M) a modified cotangent bundle; if $\omega = 0$ we call it canonical. In step 2 one realizes any modified cotangent bundle as a reduction of a canonical cotangent

bundle and in step 3 one realizes a canonical cotangent bundle as a reduction of some standard \mathbf{R}^{2n} . One then invokes the transitivity of reduction [GT] to consolidate these three steps. We will show that with the assumptions on the group G , we can tag along the G -action.

We start by giving the equivariant versions of steps 1 and 3; we do not provide (full) proofs because these are elementary and moreover similar to the corresponding proofs of the non-equivariant version in [GT]. Finally we prove the equivariant version of step 2.

Proposition 3 (Step 1). *Let a Lie group G act symplectically on a symplectic manifold (M, ω) . Then M can be realized as an equivariant reduction of the modified cotangent bundle $(T^*M, d\theta_M + \tau_M^*\omega)$ by the zero-section. The symplectic action of G on T^*M is just the cotangent lift of the action on M .*

Proposition 4 (Step 3). *Let a compact Lie group G act on a manifold P of finite type. Then the canonical cotangent bundle T^*P along with the cotangent action of G can be realized as an equivariant reduction of some \mathbf{R}^{2n} , with its standard symplectic structure, where the G -action on \mathbf{R}^{2n} is the cotangent lift of an orthogonal action of G on \mathbf{R}^n .*

Proof. According to the Mostow-Palais theorem ([Mo],[Pa]) the conditions stated above enable us to find an orthogonal representation space \mathbf{R}^n for G in which P can be equivariantly embedded. We then define $C = \tau_{\mathbf{R}^n}^{-1}(P) \subset T^*\mathbf{R}^n \cong \mathbf{R}^{2n}$ and it easily follows that T^*P is realized as the equivariant reduction of \mathbf{R}^{2n} by C . \square

It remains to provide an equivariant version of step 2. Since this requires a rather technical proof, we start with some preparations. If ω is a closed 2-form on a manifold M , we can define the set

$$\mathcal{P}_\omega = \{ f \in C^\infty(M) \mid \exists \text{ a vectorfield } X \text{ on } M : i(X)\omega + df = 0 \} .$$

Denote by X_f any vector field satisfying $i(X_f)\omega + df = 0$. Using these X_f , \mathcal{P}_ω can be turned into a Poisson algebra by defining

$$\{f, g\} = \omega(X_f, X_g) = X_f g .$$

Now let G be a Lie group acting smoothly on M and leaving the closed 2-form ω invariant. Denote by ξ_M the fundamental vector field on M associated to $\xi \in \mathfrak{g}$ (\mathfrak{g} the Lie algebra of G) whose flow φ_t is the action of $\exp(-\xi t)$ on M . A momentum map J for ω is a map $J : M \rightarrow \mathfrak{g}^*$ such that

$$\forall \xi \in \mathfrak{g} : i(\xi_M)\omega + dJ^*\xi = 0$$

where we view $\xi \in \mathfrak{g}$ as a (linear) function on \mathfrak{g}^* , i.e., $J^*\xi \in C^\infty(M)$. It follows that $J^*\xi \in \mathcal{P}_\omega$. We will say that J is an Ad^* -equivariant momentum map if $J^* : \mathfrak{g} \rightarrow \mathcal{P}_\omega$ is a Lie algebra morphism. If the group G is connected, this is equivalent to the characterization that J is equivariant with respect to the given action of G on M and the Ad^* -action on \mathfrak{g}^* .

We continue our preparations by recalling an elementary fact from differential geometry. Let $U(1) = \mathbf{T}$ act on the manifold M , let α be a closed $(k+1)$ -form on M which is invariant under the \mathbf{T} -action, and let γ be a k -cycle (for the homology of M with coefficients in \mathbf{Z}). If we denote by $\tilde{\gamma}$ the orbit of γ under the \mathbf{T} -action, then $\tilde{\gamma}$ is a $(k+1)$ -cycle. Finally let ξ be a generator of the \mathbf{T} -action of period 1, i.e., $\exp(\xi t) = 1 \Leftrightarrow t \in \mathbf{Z}$. Given all this, the next lemma is elementary although perhaps not standard; the minus sign comes from our convention on the fundamental vector field ξ_M .

Lemma 5. $\int_{\tilde{\gamma}} \alpha = - \int_{\gamma} i(\xi_M)\alpha$.

The next ingredient we will need is a special way to decompose a given closed 2-form.

Lemma 6. *Let ω be a closed 2-form on a manifold M of finite type. Let G be a compact and connected Lie group acting smoothly on M and leaving ω invariant. Then there exists a finite number of G -invariant integral closed 2-forms c_1, \dots, c_r and real numbers μ_j such that $\omega = \sum_j \mu_j \cdot c_j$.*

Moreover, if ω admits a momentum map, then each c_j also admits a momentum map. Even stronger, each c_j admits an Ad^ -equivariant momentum map.*

Proof. Since M is of finite type, the universal coefficient theorem implies that there exist a finite number of integral closed 2-forms c_1, \dots, c_r and constants $\mu_j \in \mathbf{R}$ such that $\omega = \sum_j \mu_j \cdot c_j$. Since G is compact and acts smoothly on M , we can form the averages \bar{c}_j of the c_j :

$$\bar{c}_j = \int_G g^* c_j dg$$

where dg is an invariant Haar measure on G of total volume 1. Obviously the \bar{c}_j are G -invariant, and $\omega = \sum_j \mu_j \cdot \bar{c}_j$ because ω is G -invariant. If γ is a 2-cycle on M , then for $g \in G$ we have: $\int_{\gamma} g^* c_j = \int_{g \cdot \gamma} c_j \equiv n_j(g) \in \mathbf{Z}$ because c_j is integral. Since G acts smoothly, $n_j(g)$ is a smooth function on G and hence is constant on the connected group G . It follows that $\int_{\gamma} \bar{c}_j = \int_{\gamma} c_j$ for all 2-cycles γ and hence by de Rham duality we find that the cohomology class of \bar{c}_j is the same as that of c_j . In particular \bar{c}_j is integral. This proves the first assertion.

Next we assume that the number r of c_j 's involved is minimal. It then follows that the real numbers μ_j are independent over \mathbf{Q} . To see this, suppose $\mu_r = \sum_{j=1}^{r-1} p_j \cdot \mu_j / q_j$ with $p_j, q_j \in \mathbf{Z}, q_j \neq 0$, then $\omega = \sum_{j=1}^{r-1} (\mu_j / q_j) \cdot (q_j \cdot c_j + p_j \cdot c_r)$, contradicting the minimality of r .

Now suppose ω admits a momentum map, i.e., for all $\xi \in \mathfrak{g} : i(\xi_M)\omega$ is exact. Using de Rham duality, this is equivalent to saying that for all 1-cycles γ we have $\int_\gamma i(\xi_M)\omega = 0$. The argument to prove that each c_j admits a momentum map breaks into two parts. The structure theory of compact groups tells us that there exists a finite cover of G which is the direct product of a torus \mathbf{T}^d and a simply connected semi-simple compact group H . It follows that the Lie algebra \mathfrak{g} of G is the direct sum $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$, with \mathfrak{a} the abelian Lie algebra of \mathbf{T}^d and \mathfrak{h} the semi-simple Lie algebra of H . Since \mathfrak{h} is semi-simple, it follows from the Whitehead lemmas that there exists an Ad^* -equivariant momentum map $M \rightarrow \mathfrak{h}^*$ for each c_j separately. Now let $\xi \in \mathfrak{a}$ be a vector of period 1. We then apply lemma 5 (with γ an arbitrary 1-cycle) to the equation $i(\xi_M)\omega = \sum_j \mu_j \cdot i(\xi_M)c_j$ to obtain

$$0 = - \int_\gamma i(\xi_M)\omega = \sum_j \mu_j \cdot \left(\int_\gamma c_j \right) = \sum_j \mu_j \cdot n_j \quad (n_j \in \mathbf{Z}) .$$

Since the μ_j are independent over \mathbf{Q} , the integers n_j must be zero. Hence, again by lemma 5, we have $\int_\gamma i(\xi_M)c_j = 0$, i.e., $i(\xi_M)c_j$ is exact because γ was arbitrary. Since by the compactness of \mathbf{T}^d vectors of period 1 span \mathfrak{a} , this proves the existence of a momentum map $M \rightarrow \mathfrak{a}^*$ for each c_j separately. By combining this with with the momentum map to \mathfrak{h}^* we thus have proven the second assertion.

Finally we have to prove that the momentum map for each c_j can be chosen to be Ad^* -equivariant. We know already that there exists an Ad^* -equivariant momentum map for the \mathfrak{h} part. It remains to show that we can complete this to a momentum map $J : M \rightarrow \mathfrak{g}^*$ such that for $\xi \in \mathfrak{a}$ and $\eta \in \mathfrak{g}$ (!) we have $c_j(\xi_M, \eta_M) = J^*([\xi, \eta]) \equiv J^*0 = 0$. But this is nearly trivial, regardless of the extension of the momentum map for \mathfrak{h} to one for \mathfrak{g} . Again suppose $\xi \in \mathfrak{a}$ is of period 1, then for any 0-cycle $\{m\}$ we have

$$c_j(\eta_M, \xi_M)|_m = i(\xi_M)(i(\eta_M)c_j)|_m = - \int_{\{\bar{m}\}} i(\eta_M)c_j = 0$$

where we use that $i(\eta_M)c_j$ is invariant under the flow of ξ_M (ξ commutes with η) and that $i(\eta_M)c_j$ is exact (there exists a momentum map). This proves the last assertion. \square

We now come to the basic ingredient for step 2 : the prequantization construction. Given a closed 2-form ω on a manifold M which represents an integral cohomology class, there exists a principal \mathbf{T} -bundle P over M with a connection α whose curvature is exactly ω . Such a principal bundle (P, α) is called a *prequantization* of (M, ω) . To incorporate a group action we need the following lemma.

Lemma 7. *Let a compact connected Lie group G act on M preserving ω . Then there exists an integer k depending only on G such that the action of G can be lifted to a*

connection preserving action on any prequantization (P, α) of $(M, k\omega)$ if and only if the G -action admits an Ad^* -equivariant momentum map.

Proof. A proof of this lemma based on spectral sequence arguments can be found in [On]; we present here a more elementary proof using the theory of central extensions developed in [TW].

First note that if the G -action can be lifted to a prequantization of $(M, k\omega)$ for some $k \in \mathbf{Z}$, then there exists an Ad^* -equivariant momentum map (given by contracting the fundamental vector fields of the action on P with the connection form α).

On the other hand, as in the proof of lemma 6, there exists a finite cover $\mathbf{T}^d \times H \rightarrow G$ with H simply connected and compact. Since $\mathbf{T}^d \times H$ is a cover of G , it also acts on M and admits the same Ad^* -equivariant momentum map. The existence of this momentum map guarantees that there exists a connection preserving action of a central extension E of $\mathbf{T}^d \times H$ by \mathbf{T} on any prequantization (P, α) of (M, ω) . The fact that the momentum map is Ad^* -equivariant implies that this extension is trivial on the level of Lie algebras. Finally the classification of the different possible group extensions for a given algebra extension tells us that for $\mathbf{T}^d \times H$ the extension is unique. Hence this extension E must be the trivial one $E = \mathbf{T}^d \times H \times \mathbf{T}$, so we obtain a connection preserving action of $\mathbf{T}^d \times H$ on any prequantization (P, α) of (M, ω) . Now denote by K the kernel of the cover $\mathbf{T}^d \times H \rightarrow G$ and define $k = \#K$. Since the action of K on P covers the identity transformation on M and preserves the connection, this action coincides with the action of a (finite) subgroup of the torus \mathbf{T} . Now the manifold $P/(\mathbf{Z}/k\mathbf{Z})$ (where $\mathbf{Z}/k\mathbf{Z}$ has to be seen as a subgroup of \mathbf{T}) is a prequantization of $(M, k\omega)$. (This process is called by Souriau “quantization by fusion” [So].) Moreover, by construction the induced action of $\mathbf{T}^d \times H$ is trivial for elements of K and hence it defines an action of G itself on $P/(\mathbf{Z}/k\mathbf{Z})$. Since every prequantization of $(M, k\omega)$ can be obtained in this way from a prequantization of (M, ω) we have proven the lemma. \square

With these preparations we can proceed with the equivariant version of step 2. Let ω be a closed 2-form on the manifold M of finite type and let G be a compact, connected Lie group acting on M leaving ω invariant and admitting a momentum map. We apply lemma 6 to find integral closed 2-forms c_1, \dots, c_r and constants μ_j such that $\omega = \sum_j (\mu_j/k) \cdot (kc_j)$, where k is defined as in the proof of lemma 7. If we denote by (P_j, α_j) any prequantization of (M, kc_j) , then we can form the product bundle $P = \prod_M P_j$ as the principal \mathbf{T}^r -bundle over M with connection $\alpha = (\alpha_1, \dots, \alpha_r)$ whose curvature is (kc_1, \dots, kc_r) .

According to lemma 6 we see that the kc_j satisfy the conditions of lemma 7. As a consequence, the G -action on M lifts to a connection preserving action on P and hence commutes with the \mathbf{T}^r -action. We now invoke the cotangent bundle reduction theorem ([Ku],[MR]) to prove in the first place that $(T^*M, d\theta_M + \tau_M^* \omega)$ can be realized as the reduc-

tion of the canonical cotangent bundle T^*P by the submanifold $C = J^{-1}(\mu_1/k, \dots, \mu_r/k)$, where J is the canonical momentum map for the lifted \mathbf{T}^r -action on T^*P . In the second place, if we lift the G -action on P canonically to T^*P , then a careful analysis of the proof of the cotangent bundle reduction theorem (using the facts that the G -action preserves the connection and commutes with the \mathbf{T}^r -action) shows that this reduction is actually an equivariant reduction. This discussion proves the next proposition, which is the equivariant version of step 2.

Proposition 8 (Step 2). *Let ω be a closed 2-form on the manifold M of finite type and let G be a compact, connected Lie group acting on M , leaving ω invariant and admitting a momentum map. The cotangent action of G on T^*M is a symplectic action for the modified symplectic form $d\theta_M + \tau_M^*\omega$, and we can realize $(T^*M, d\theta_M + \tau_M^*\omega, G)$ as an equivariant reduction of some canonical cotangent bundle T^*P , where the action of G on T^*P is the canonically lifted action.*

It is routine to verify that equivariant reduction is transitive. Combining this observation with propositions 3, 8 and 4 consecutively proves Theorem 1.

Proof of Proposition 2.

First we consider the following slightly more general situation. Suppose that $G = H \times K$ with K compact acts on M leaving ω invariant, and suppose that the H -action admits a momentum map J . Since the actions of H and K commute, the average \bar{J} of J over K is also a momentum map for H , one which is moreover invariant under the action of K . Since two momentum maps differ by a locally constant function on M , it follows that J is infinitesimally invariant under the action of K . In particular, if K or M is connected, J is globally invariant under the K action.

We now return to the assumptions of the proposition and note again that the connected component of a compact group is up to a finite cover the direct product of a semi simple group and circles \mathbf{T}^1 . Since the G -action admits a momentum map, all the separate factors of this cover admit momentum maps. But, a semi-simple group always admits an Ad^* -equivariant momentum map, and a momentum map for a circle action is always Ad^* -equivariant. Hence the various factors admit Ad^* -equivariant momentum maps. Next we note that if we combine the Ad^* -equivariant momentum maps of the various factors into one momentum map for (the cover of) G , a necessary and sufficient condition for the combination also to be Ad^* -equivariant is that the momentum map of each factor is infinitesimally invariant under the action of the other factors. But this follows from our averaging argument above because all the separate factors are compact. Thus we have proven Proposition 2.

Remark. If we assume in Proposition 2 that M is of finite type, then it is an immediate corollary of lemma 6 by the following argument. Let J_j be Ad^* -equivariant momentum maps for the c_j according to lemma 6, then $J = \sum \mu_j \cdot J_j$ is an Ad^* -equivariant momentum map for ω .

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