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Polynomial algebras on coadjoint orbits of semisimple Lie groups

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Abstract

We study the algebraic structure of the Poisson algebra $P(\mathcal{O})$ of polynomials on a coadjoint orbit $\mathcal{O}$ of a semisimple Lie algebra. We prove that $P(\mathcal{O})$ splits into a direct sum of its Lie center and its derived Lie ideal. We also show that $P(\mathcal{O})$ is simple as a Poisson algebra iff $\mathcal{O}$ is semisimple. © 2002 Elsevier Science B.V. All rights reserved.

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1. Structure theorems

Let $\mathfrak{g}$ be a real (finite-dimensional) semisimple Lie algebra with corresponding 1-connected Lie group G. It is well known that the dual space $\mathfrak{g}^*$ carries the structure of a linear Poisson manifold under the Lie–Poisson bracket. The symplectic leaves of this Poisson structure are the orbits of the coadjoint representation of G on $\mathfrak{g}^*$.

As the elements of $\mathfrak{g}$ may be regarded as linear functions on $\mathfrak{g}^*$, the symmetric algebra $S(\mathfrak{g})$ may be identified with the algebra of polynomial functions on $\mathfrak{g}^*$. Consequently, $S(\mathfrak{g})$ can be realized as a Poisson subalgebra of $C^\infty(\mathfrak{g}^*)$. (Equivalently, the

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Poisson bracket $\{\ ,\ \}$ on $S(\mathfrak{g})$ can be obtained by setting $\{\xi,\eta\}=[\xi,\eta]$ for $\xi,\eta\in\mathfrak{g}$ and extending to all of $S(\mathfrak{g})$ via the Leibniz rule.)

Let $S(\mathfrak{g})'=\{S(\mathfrak{g}),S(\mathfrak{g})\}$ be the derived Lie ideal, and let $C(\mathfrak{g})$ denote the Lie center of the Poisson algebra $S(\mathfrak{g})$; elements of $C(\mathfrak{g})$ are called "Casimirs".

Proposition 1. $S(\mathfrak{g})=C(\mathfrak{g})\oplus S(\mathfrak{g})'$.

Proof. We have the decomposition

$$S(\mathfrak{g})=\bigoplus_{n=0}^{\infty}S_n(\mathfrak{g})$$

of the symmetric algebra into the finite-dimensional subspaces $S_n(\mathfrak{g})$ of homogeneous elements of degree n. Each $S_n(\mathfrak{g})$ is invariant with respect to the adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$. Since every finite-dimensional representation of a semisimple Lie algebra is completely reducible, it follows that the adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$ is itself completely reducible. According to [2, Section 1.2.10] we can then split

$$S(\mathfrak{g})=C(\mathfrak{g})\oplus\{\mathfrak{g},S(\mathfrak{g})\}.$$

So we need only show that $\{\mathfrak{g},S(\mathfrak{g})\}=S(\mathfrak{g})'$.

Now, applying the identity

$$\{fg,h\}=\{f,gh\}+\{g,fh\}$$

to $f,g\in\mathfrak{g}$ and $h\in S_n(\mathfrak{g})$, we see that $\{S_2(\mathfrak{g}),S_n(\mathfrak{g})\}\subset\{\mathfrak{g},S_{n+1}(\mathfrak{g})\}$. Arguing recursively, we obtain

$$\{S_m(\mathfrak{g}),S_n(\mathfrak{g})\}\subset\{\mathfrak{g},S_{n+m-1}(\mathfrak{g})\},$$

from which the desired result follows. □

Let $\mathcal{O}$ be an orbit in $\mathfrak{g}^*$. We can restrict polynomials on $\mathfrak{g}^*$ to functions on $\mathcal{O}$ thereby obtaining the (orbit) polynomial algebra $P(\mathcal{O})$ (which, however, may not be freely generated as an associative algebra). We may identify $P(\mathcal{O})$ with the quotient algebra $S(\mathfrak{g})/I(\mathcal{O})$, where $I(\mathcal{O})$ is the associative ideal of elements vanishing on $\mathcal{O}$, with the canonical projection

$$\rho_{\mathcal{O}}:S(\mathfrak{g})\to S(\mathfrak{g})/I(\mathcal{O})\cong P(\mathcal{O}).$$

Since $\mathcal{O}$ is a symplectic leaf of the Poisson structure on $\mathfrak{g}^*$, $I(\mathcal{O})$ is a Lie ideal as well. Thus $I(\mathcal{O})$ is a Poisson ideal (i.e., an associative ideal which is also a Lie ideal) and hence $P(\mathcal{O})$ is a Poisson algebra of polynomial functions on the symplectic leaf $\mathcal{O}$. Note that since $\mathcal{O}$ is symplectic, the Lie center $Z(P(\mathcal{O}))=\mathbb{R}$.

We now show that the decomposition in Proposition 1 projects to a similar decomposition of $P(\mathcal{O})$.

Theorem 2. $P(\mathcal{O})=\mathbb{R}\oplus P(\mathcal{O})'$.

Proof. It is clear that $C(\mathfrak{g})$ projects onto constants on $\mathcal{O}$ and $\rho_{\mathcal{O}}(S(\mathfrak{g})') = P(\mathcal{O})'$, so that $\mathbb{R} + P(\mathcal{O})' = P(\mathcal{O})$ by Proposition 1. It remains to show that $P(\mathcal{O})' \cap \mathbb{R} = \{0\}$.

Now the restriction of the adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$ to the invariant subspace $I(\mathcal{O})$ is also completely reducible, so we can again use [2, Section 1.2.10] to split $I(\mathcal{O}) = I(\mathcal{O})_1 \oplus I(\mathcal{O})_2$, where $I(\mathcal{O})_1 = I(\mathcal{O}) \cap C(\mathfrak{g})$ and $I(\mathcal{O})_2 = \{\mathfrak{g}, I(\mathcal{O})\} \subset S(\mathfrak{g})'$.

If $P(\mathcal{O})' \cap \mathbb{R} \neq \{0\}$, then there is an $f \in I(\mathcal{O})$ such that $1 + f \in S(\mathfrak{g})'$. Decomposing $f = f_1 + f_2$ with $f_1 \in I(\mathcal{O})_1$ and $f_2 \in I(\mathcal{O})_2$, we get $(1 + f_1) + f_2 \in S(\mathfrak{g})'$, so $1 + f_1 \in C(\mathfrak{g}) \cap S(\mathfrak{g})' = \{0\}$. Hence $f_1 = -1$, and this contradicts the fact that $f_1 \in I(\mathcal{O})_1 \subset I(\mathcal{O})$. □

Theorem 2 was already known in the case when $\mathcal{O}$ is compact [3]. In the C^∞ context, one knows that if M is a compact symplectic manifold, then its Poisson algebra [1]

$$C^\infty(M) = \mathbb{R} \oplus C^\infty(M)',$$

while if M is noncompact [7]

$$C^\infty(M) = C^\infty(M)'.$$

Since in the smooth case $f \in C^\infty(M)'$ if and only if $f\eta$ is an exact form, where η is the Liouville volume form, Theorem 2 thus suggests that the polynomial Poisson (resp. de Rham) cohomology of a noncompact coadjoint orbit $\mathcal{O}$ may differ from its smooth Poisson (resp. de Rham) cohomology. For example, take $\mathcal{O} \subset sl(2,\mathbb{R})^*$ to be the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$. The Poisson tensor

$$\Lambda = x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x - z\partial_x \wedge \partial_y$$

on $sl(2,\mathbb{R})^*$ is polynomial and the induced symplectic form

$$\omega = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y$$

on $\mathcal{O}$ is also polynomial. As ω is a volume form on the noncompact manifold $\mathcal{O}$, it is exact in the smooth category. It is, however, not exact in the polynomial category. Indeed, if $\omega = \mathrm{d}\alpha$ for some polynomial 1-form α on $\mathcal{O}$, then, according to the well-known isomorphism between Poisson and de Rham cohomology on a symplectic manifold, we would have $[\Lambda, i_\alpha \Lambda] = \Lambda$, where $[\ ,\ ]$ is the Schouten bracket and $[\Lambda, \cdot]$ is the Poisson cohomology differential [5,8]. Writing $\alpha = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$, where f, g, h are polynomials, this gives

$$\Lambda = H_x \wedge H_f + H_y \wedge H_g + H_z \wedge H_h,$$

where H_a is the Hamiltonian vector field of a. Contracting Λ with ω then yields $1 = \{x, f\} + \{y, g\} + \{z, h\}$—a contradiction with Theorem 2.

We remark that Theorem 2 need not hold if $\mathfrak{g}$ is not semisimple. For instance, $\mathbb{R}^{2n}$ with its standard symplectic structure is a coadjoint orbit of the Heisenberg group $\mathrm{H}(2n)$, but in this case $P(\mathbb{R}^{2n}) = P(\mathbb{R}^{2n})'$.

2. A characterization of $P(\mathcal{O})$

We call a Lie algebra L *essentially simple* if every Lie ideal of L is either contained in the Lie center $Z(L)$ of L or contains the derived Lie ideal $L' = [L, L]$. We say that a Poisson algebra P is *simple* if the only Poisson ideals of P are P and $\{0\}$, and *unital* if it contains an associative identity.

Proposition 3. *Let P be a unital Poisson algebra which has no nilpotent elements with respect to the associative structure. If P is simple, then it is essentially simple.*

Proof. In view of [4, Theorem 1.10], if L is a Lie ideal of a unital Poisson algebra P then

$$\{P, ad^{-1}(L)\} \subset r(J(L)), \tag{1}$$

where

$$ad^{-1}(L) = \{f \mid \{f, P\} \subset L\},$$

$J(L)$ is the largest associative ideal of P contained in $ad^{-1}(L)$, and $r(J(L))$ is its radical,

$$r(J(L)) = \{f \mid f^n \in J(L) \text{ for some } n = 1, 2, \ldots\}.$$

We recall from [4, Theorem 1.6] that $J(L)$ is in fact a Poisson ideal of P.

Suppose that $P' \not\subset L$. Then $ad^{-1}(L) \neq P$, so $J(L) \neq P$, and thus $J(L) = \{0\}$ as P is simple. Then $r(J(L)) = \{0\}$ since by assumption P has no associative nilpotents. Then (1) gives

$$\{P, L\} \subset \{P, ad^{-1}(L)\} = \{0\},$$

i.e., $L \subset Z(P)$. □

In particular, the hypotheses of Proposition 3 are satisfied by the polynomial algebra $P(\mathcal{O})$. We now use this proposition to prove our main result. Recall that $\mathcal{O}$ is said to be *semisimple* (resp. *nilpotent*) if it is the orbit of a semisimple (resp. nilpotent) element of $\mathfrak{g}^*$.

Theorem 4. *The Lie algebra $P(\mathcal{O})$ is essentially simple iff the orbit $\mathcal{O}$ is semi-simple.*

Proof. ($\Leftarrow$) Assume $\mathcal{O}$ is semisimple and let $\mathcal{O}_{\mathbb{C}}$ be the complexification of $\mathcal{O}$, i.e., the orbit in $\mathfrak{g}^*_{\mathbb{C}}$ with respect to the complexified Lie group $G_{\mathbb{C}}$ which contains $\mathcal{O}$ in its real part. It is well known that $\mathcal{O}_{\mathbb{C}}$ is semisimple and that $\mathcal{O}_{\mathbb{C}}$ is an algebraic set in $\mathfrak{g}^*_{\mathbb{C}}$ [6, Section 3.8]. If $P(\mathcal{O})$ were not essentially simple, then by Proposition 3 we would have a proper Poisson ideal I in $P(\mathcal{O})$, and so, after complexification, a proper Poisson ideal $I_{\mathbb{C}}$ in $P(\mathcal{O})_{\mathbb{C}} := P_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}})$.

Let $V(I_{\mathbb{C}})$ be the set of zeros of $I_{\mathbb{C}}$ in $\mathcal{O}_{\mathbb{C}}$. Since $\mathcal{O}_{\mathbb{C}}$ is algebraic, $V(I_{\mathbb{C}}) \neq \emptyset$, and since $I_{\mathbb{C}}$ is a Lie ideal, $V(I_{\mathbb{C}})$ is $G_{\mathbb{C}}$-invariant and hence consists of orbits. This forces $V(I_{\mathbb{C}}) = \mathcal{O}_{\mathbb{C}}$ and so $I_{\mathbb{C}} = \{0\}$. Hence we have a contradiction, since $I_{\mathbb{C}}$ is proper.

($\Rightarrow$) Assume that $\mathcal{O}$ is not semisimple. Complexifying as before, we get the complexified orbit $\mathcal{O}_{\mathbb{C}}$ which is not semisimple. Now there exists a semisimple orbit $\mathcal{S}$ in the Zariski closure of $\mathcal{O}_{\mathbb{C}}$ [6, Section 3.8]. Consider the Poisson ideal K of elements of $P(\mathcal{O})$ which vanish on $\mathcal{S}$. We claim that this ideal is proper. Indeed, $K=\{0\}$ implies that $I(\mathcal{S})=I(\mathcal{O}_{\mathbb{C}})$, whence $\mathcal{S}=\mathrm{cl}(\mathcal{O}_{\mathbb{C}})$ as $\mathcal{S}$ is an algebraic set. But this is impossible as $\mathcal{S}$ and $\mathcal{O}_{\mathbb{C}}$ are distinct orbits. As well, $K=P(\mathcal{O})_{\mathbb{C}}$ is impossible as $\mathcal{S}\neq\emptyset$.

Now, we will show that the existence of the proper Poisson ideal K in the complex Poisson algebra $P(\mathcal{O})_{\mathbb{C}}$ implies the existence of a proper Poisson ideal I in $P(\mathcal{O})$. First, put

$$K_{\mathbb{R}}=\{f\in P(\mathcal{O})\mid f+\mathrm{i}g\in K \text{ for some } g\in P(\mathcal{O})\}.$$

Since for $h\in P(\mathcal{O})$, $f+\mathrm{i}g\in K$ implies $(hf)+\mathrm{i}(hg)\in K$ and $\{h,f\}+\mathrm{i}\{h,g\}\in K$, $K_{\mathbb{R}}$ is a Poisson ideal of $P(\mathcal{O})$. Clearly $K\subset K_{\mathbb{R}}+\mathrm{i}K_{\mathbb{R}}$ so that if $K_{\mathbb{R}}=\{0\}$ then $K=\{0\}$. We can thus take $I=K_{\mathbb{R}}$ as long as $K_{\mathbb{R}}\neq P(\mathcal{O})$.

If $K_{\mathbb{R}}=P(\mathcal{O})$, then there is $g\in P(\mathcal{O})$ such that $1+\mathrm{i}g\in K$. Let

$$K_0=\{f\in P(\mathcal{O})\mid \mathrm{i}f\in K\}.$$

Similarly as for $K_{\mathbb{R}}$, we can prove that K_0 is a Poisson ideal. Now $K_0\neq P(\mathcal{O})$, for otherwise $K=P(\mathcal{O})_{\mathbb{C}}$. We can then take $I=K_0$ provided $K_0\neq\{0\}$. But in fact $K_0\neq\{0\}$: Since

$$\{P(\mathcal{O}),1+\mathrm{i}g\}=\mathrm{i}\{P(\mathcal{O}),g\}\subset K,$$

$\{P(\mathcal{O}),g\}\subset K_0$, and so $K_0=\{0\}$ implies that $g\in Z(P(\mathcal{O}))=\mathbb{R}$. So $1+\mathrm{i}g\in K$ is a nonzero constant, whence again $K=P(\mathcal{O})_{\mathbb{C}}$.

In any eventuality, we now have a proper Poisson ideal I of $P(\mathcal{O})$. Of course, $I\not\subset Z(P(\mathcal{O}))=\mathbb{R}$. However, it may happen that $I\supset P(\mathcal{O})'$, in which case Theorem 2 forces $I=P(\mathcal{O})'$. In this circumstance we pass to the associative ideal I^2. Since

$$\{P(\mathcal{O}),I^2\}\subset\{P(\mathcal{O}),I\}I\subset I^2,$$

I^2 is also a Lie ideal. If $I^2\neq I$, then I^2 is a proper Lie ideal which neither is contained in the Lie center nor contains the derived Lie ideal.

To see that $I^2\neq I$ for I proper, we can use the following.

Lemma 1. *If P is a commutative unital ring with no zero divisors and I is a proper ideal which is finitely generated, then $I^2\neq I$.*

Proof. Assume that $x_1,\ldots,x_n$ are generators of I and $I^2=I$. Then $x_i=\sum_{j=1}^n a_{ij}x_j$ for some $a_{ij}\in I$, so that $\sum_{j=1}^n b_{ij}x_j=0$ where $b_{ij}=\delta_{ij}-a_{ij}$. Setting $B=(b_{ij})$, Cramer's rule gives $x_i\det B=0$ whence $\det B=0$. But $\det B\in\{1\}+I$ so $\det B\neq 0$.

Thus $P(\mathcal{O})$ is not essentially simple. □

The last part of this proof provides a converse to Proposition 3 when $P=P(\mathcal{O})$. In particular, we conclude that $P(\mathcal{O})$ is simple if and only if $\mathcal{O}$ is semisimple.

One can also see explicitly that $P(\mathcal{O})$ is not essentially simple when $\mathcal{O}$ is nilpotent as follows. Since a nilpotent orbit is conical, $I(\mathcal{O})$ is a homogeneous associative ideal. As a consequence, the notion of homogeneous polynomial makes sense in $P(\mathcal{O})$. Let $P_k(\mathcal{O})$ denote the subspace consisting of all homogeneous polynomials of degree k. By virtue of the commutation relations of $\mathfrak{g}$,

$$\{P_k(\mathcal{O}), P_l(\mathcal{O})\} \subset P_{k+l-1}(\mathcal{O}),$$

whence each $P_{(k)}(\mathcal{O}) = \bigoplus_{\ell \geqslant k} P_\ell(\mathcal{O})$ for $k \geqslant 1$ is a proper Poisson ideal of $P(\mathcal{O})$.

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