Problem 4.5.8. Let us say the life of a tire in miles, say $X$, is normally distributed with mean $\theta$ and standard deviation 5000. Past experience indicates that $\theta = 30,000$. The manufacturer claims that the tires made by a new process have a mean $\theta > 30,000$. It is possible that $\theta = 35,000$. Check his claim by testing $H_0 : \theta = 30,000$ against $H_1 : \theta > 30,000$. We observe $n$ independent values of $X$, say $x_1, \ldots, x_n$, and we reject $H_0$ (thus accept $H_1$) if and only if $\bar{x} \geq c$. Determine $n$ and $c$ so that the power function $\gamma(\theta)$ of the test has the values $\gamma(30,000) = 0.01$ and $\gamma(35,000) = 0.98$.

Solution 4.5.8. We are assuming that $X$ has a normal distribution with unknown mean $\theta$ and standard deviation $\sigma = 5000$ miles. Therefore, the sample mean $\bar{X} = \left(\frac{5000}{\sqrt{n}}\right)Z + \theta$, where $Z$ has the standard normal distribution. Since the form of the rejection region is the interval $(c, \infty)$, the power function is
\[
\gamma(\theta) = P(\bar{X} > c) = P\left(\frac{5000}{\sqrt{n}} Z + \theta > c\right) = P\left(Z > \frac{\sqrt{n}(c - \theta)}{5000}\right).
\]
The equations $\gamma(30,000) = 0.01$ and $\gamma(35,000) = 0.98$ yield two equations for $n$ and $c$,
\[
(1) \quad \frac{\sqrt{n}(c - 30000)}{5000} = z_{0.01}
\]
\[
(2) \quad \frac{\sqrt{n}(c - 35000)}{5000} = z_{0.98} = -z_{0.02}.
\]
Subtracting equation (2) from equation (1) eliminates the constant $c$, and simplification leaves
\[
\sqrt{n} = z_{0.01} + z_{0.02} \approx 2.3263 + 2.0537 = 4.3800.
\]
Thus, to have power at least 0.98 for $\theta = 35000$ the sample size must be at least $n = \lceil (4.3800)^2 \rceil = \lceil 19.1844 \rceil = 20$. Substituting $n = 20$ into equation (1) and solving for $c$ we find $c = 30000 + \left(\frac{5000}{\sqrt{20}}\right)z_{0.01} \approx 30000 + 2600.936 \approx 32601$. Therefore, for the significance level of the test to be at most $\alpha = 0.01$ and for the power of the test at $\theta = 35000$ miles to be at least 0.98 we should take $n = 20$, $c = 32601$.

Problem 4.5.9. Let $X$ have a Poisson distribution with mean $\theta$. Consider the simple hypothesis $H_0 : \theta = 1/2$ and the alternative composite hypothesis $H_1 : \theta < 1/2$. Thus $\Omega = \{\theta : 0 < \theta \leq 1/2\}$. Let $X_1, \ldots, X_{12}$ denote a random sample of size 12 from this distribution. We reject $H_0$ if and only if the observed value of $Y = X_1 + \cdots + X_{12} \leq 2$. If $\gamma(\theta)$ is the power function
of the test, find the powers $\gamma(1/2)$, $\gamma(1/3)$, $\gamma(1/4)$, $\gamma(1/6)$, and $\gamma(1/12)$. Sketch the graph of $\gamma(\theta)$. What is the significance level of the test?

**Solution 4.5.9.** By Theorem 3.2.1 on page 153 we see that $Y$ has a Poisson distribution with mean $12\theta$. Thus we have an exact expression for the power function

$$\gamma(\theta) = P(Y \leq 2) = \sum_{y=0}^{2} e^{-12\theta} \frac{(12\theta)^y}{y!} = (1 + 12\theta + 72\theta^2) e^{-12\theta},$$

for $0 < \theta \leq 1/2$. The significance level is $\alpha = \gamma(1/2) = 0.062$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$1/2$</th>
<th>$1/3$</th>
<th>$1/4$</th>
<th>$1/6$</th>
<th>$1/12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma(\theta)$</td>
<td>0.06198</td>
<td>0.2382</td>
<td>0.4232</td>
<td>0.6765</td>
<td>0.9198</td>
</tr>
</tbody>
</table>

**Table 1.** Approximate Values of the Power Function $\gamma(\theta)$

**Power Function**

![Graph of the Power Function $\gamma(\theta) = P_\theta(Y \leq 2)$](image)

**Figure 1.** Graph of the Power Function $\gamma(\theta) = P_\theta(Y \leq 2)$
Problem 4.5.10. Let $Y$ have a binomial distribution with parameters $n$ and $p$. We reject $H_0 : p = 1/2$ and accept $H_1 : p > 1/2$ if $Y \geq c$. Find $n$ and $c$ to give a power function $\gamma(p)$ which is such that $\gamma(1/2) = 0.10$ and $\gamma(2/3) = 0.95$, approximately.

Solution 4.5.10. We will approximate $Y$ by a normally distributed random variable that we will call $X$. Therefore, $Y \approx X = \sqrt{np(1-p)}Z + np$, where $Z$ has the standard normal distribution. Since the form of the rejection region is the interval $[c, \infty)$, the power function is $$\gamma(p) = P(Y \geq c) \approx P(X > c) = P\left(Z > \frac{c - np}{\sqrt{np(1-p)}}\right).$$

The equations $\gamma(1/2) = 0.10$ and $\gamma(2/3) = 0.95$ yield two approximate equations for $n$ and $c$,

$$\frac{2c - n}{\sqrt{n}} = z_{0.10}$$

$$\frac{3c - 2n}{\sqrt{2n}} = z_{0.95} = -z_{0.05}.$$  

Multiplying equation (4) by $2\sqrt{2}$ and subtracting the result from 3 times equation (3) eliminates the constant $c$, and simplification leaves

$$\sqrt{n} = 3z_{0.10} + 2\sqrt{2}z_{0.05} = 3.8447 + 4.6523 = 8.4970.$$ 

Thus, to have approximate power 0.95 for $p = 2/3$ the sample size should be about $n = \lceil (8.4970)^2 \rceil = \lceil 72.1990 \rceil = 73$. Substituting $n = 73$ into equation (3) and solving for $c$ we find $c = \lceil (n + \sqrt{n}z_{0.10})/2 \rceil = \lceil 41.9748 \rceil = 42$. So, using the normal approximation to the binomial we arrive at approximate values of $n \approx 73$ and $c \approx 42$. Having these values, we can use the binomial CDF function in our calculator to get a better idea of the true values. Using $n = 73$ and $c = 42$ we find $\alpha = \gamma(1/2) = P(Y \geq 42 \mid p = 1/2) \approx 0.1208$ and $\gamma(2/3) = P(Y \geq 42 \mid p = 2/3) \approx 0.9604$. While the power at $p = 2/3$ is more than 0.95 (which is good) the significance level ($\alpha = P(\text{Type I Error}) = \gamma(1/2)$) is a little more than we want. Let us see what happens when we increase the value of $c$ from 42 to 43 and when we increase the value of $n$ from 73 to 74. With those changes we find $\gamma(1/2) \approx 0.1003$ and $\gamma(2/3) \approx 0.9520$. These values are much closer to what we want, so we would probably prefer to take $n = 74$ and $c = 43$.

Problem 4.5.11. Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with pdf $f(x; \theta) = 1/\theta$, $0 < x < \theta$,
zero elsewhere, where $0 < \theta$. The hypothesis $H_0 : \theta = 1$ is rejected and $H_1 : \theta > 1$ is accepted if the observed $Y_4 \geq c$.

(a) Find the constant $c$ so that the significance level is $\alpha = 0.05$.
(b) Determine the power function of the test.

Solution 4.5.11.

(a) Since $f(x; \theta)$ is the density function for a continuous uniform random variable $X$ on the interval $(0, \theta)$ we know the CDF for $X$ is $F_X(x; \theta) = x/\theta$ for $0 < x < \theta$, 0 for $x \leq 0$, and 1 for $x \geq \theta$. Since $Y_4 = \max(X_1, X_2, X_3, X_4)$ is the maximum order statistic for a random sample taken from the distribution of $X$, we know that the CDF for $Y_4$ is $F(Y_4; \theta) = [F_X(y; \theta)]^4$. This allows us to calculate

$$\alpha = P(Y_4 \geq c \mid \theta = 1) = 1 - F(c; 1) = 1 - c^4$$

for $0 < c < 1$. Therefore, $\alpha = 0.05$ if we choose $c = (0.95)^{1/4} \approx 0.98726$.

(b) The power function for the test is

$$\gamma(\theta) = P(Y_4 \geq (0.95)^{1/4}; \theta) = 1 - F((0.95)^{1/4}; \theta)$$

$$= 1 - [(0.95)^{1/4}/\theta]^4$$

$$= 1 - \frac{0.95}{\theta^4}.$$

Problem 4.5.12. Let $X_1, X_2, \ldots, X_8$ be a random sample of size $n = 8$ from a Poisson distribution with mean $\mu$. Reject the simple null hypothesis $H_0 : \mu = 0.5$ and accept $H_1 : \mu > 0.5$ if the observed sum $\sum_{i=1}^{8} x_i \geq 8$.

(a) Compute the significance level $\alpha$ of the test.
(b) Find the power function $\gamma(\mu)$ of the test as a sum of Poisson probabilities.
(c) Using Table I of Appendix C, determine $\gamma(0.75)$, $\gamma(1)$, and $\gamma(1.25)$.

Solution 4.5.12.

(a) By Theorem 3.2.1 on page 153 we see that $Y = \sum_{i=1}^{8} X_i$ has a Poisson distribution with mean $m = 8\mu$. If we assume that the null hypothesis is true, then the mean of $Y$ is $m = (8)(0.5) = 4$. Using Table I on page 656 with $m = 4$ we find that

$$\alpha = P(Y \geq 8 \mid m = 4) = 1 - P(Y \leq 7 \mid m = 4) \approx 1 - 0.949 = 0.051.$$

(b) The power function is

$$\gamma(\mu) = P(Y \geq 8 \mid m = 8\mu) = \sum_{y=8}^{\infty} e^{-8\mu}(8\mu)^y = 1 - \sum_{y=0}^{7} e^{-8\mu}(8\mu)^y.$$

(c) Using Table I on page 656

$$\gamma(0.75) = 1 - P(Y \leq 7 \mid m = 6) \approx 1 - 0.744 = 0.266.$$

$$\gamma(1.00) = 1 - P(Y \leq 7 \mid m = 8) \approx 1 - 0.453 = 0.547.$$

$$\gamma(1.25) = 1 - P(Y \leq 7 \mid m = 10) \approx 1 - 0.220 = 0.780.$$