Solutions to Homework Assignment 1

1. Find the general solution.

(b) $\frac{dx}{dt} = x(1-x)$.
Notice that there are two constant solutions: $x(t) = 0$, $x(t) = 1$. If $x \neq 0$ and $x \neq 1$, then we can separate the variables.

$$\begin{aligned}
\frac{dx}{x(1-x)} &= dt \\
\int \frac{dx}{x(1-x)}\,dx &= \int dt + \ln C, \quad \text{and using } \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x} \\
\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx &= t + \ln C \\
\ln \frac{x}{1-x} &= t + \ln C \\
\frac{x}{1-x} &= Ce^t \\
x &= Ce^t(1-x) \\
x &= \frac{Ce^t}{1+Ce^t},
\end{aligned}$$

where C can be any real number. Notice that $C = 0$ gives the constant solution $x(t) = 0$. Therefore, the general solution to this ODE is

$$x(t) = 1 \text{ or } x(t) = \frac{Ce^t}{1+Ce^t}, \quad C \in \mathbb{R}.$$

(c) $\frac{dy}{dx} = x^2y^2 + x^2 - y^2 - 1 = (x^2-1)(y^2+1)$. We can separate the variables.

$$\begin{aligned}
\frac{dy}{1+y^2} &= (x^2-1)\,dx \\
\int \frac{dy}{1+y^2} &= \int x^2 - 1\,dx + C \\
\tan^{-1} y &= \frac{1}{3}x^3 - x + C \\
y &= \tan\left(\frac{1}{3}x^3 - x + C\right).
\end{aligned}$$

(g) $\frac{dx}{dt} = te^{x+t} = te^t e^x$. We can separate the variables.

$$
\begin{aligned}
e^{-x}\,dx &= te^t\,dt \\
\int e^{-x}\,dx &= \int te^t\,dt + C_1 \\
-e^{-x} &= te^t - e^t + C_1 \\
e^x &= \frac{-1}{te^t - e^t + C_1} = \frac{1}{e^t - te^t + C}, \qquad C = -C_1 \\
x &= \ln\left(\frac{1}{e^t - te^t + C}\right) \\
x &= -\ln\left(e^t - te^t + C\right)
\end{aligned}
$$

3. Torricelli's law states that (under certain ideal circumstances) fluid will leak out of a hole at the base of a container at a rate proportional to the square root of the height of the fluid's surface from the base. Suppose that a cylindrical container is initially filled to a depth of one foot. If it takes one minute for three quarters of the fluid to leak out, how long will it take for all of the fluid to leak out?

Solution. Let $h(t)$ be the height of the fluid at time t, where t is measured in minutes. We are given that $h(0) = 1$ foot.

Let A be the area (in square feet) of the base of the cylindrical container. Then the volume of fluid in the container at time t is $V(t) = Ah(t)$ and $V(0) = A$ cubic feet.

Torricelli's law gives that $\frac{dV}{dt} = -k\,h^{1/2}$, for some positive constant k. Now, let $K = k/A^{1/2}$ and substitute $h = V/A$ into the previous ODE. This yields the following differential equation for V, which we are able to solve by separation of variables.

$$
\begin{aligned}
\frac{dV}{dt} &= -K\,V^{1/2} \\
V^{-1/2}\,dV &= -K\,dt \\
\int V^{-1/2}\,dV &= -\int K\,dt + C \\
2\,V^{1/2} &= -K\,t + C \\
V &= \frac{1}{4}\big(C - K\,t\big)^2 \\
V &= \frac{1}{4}\,C^2\big(1 - K'\,t\big)^2,
\end{aligned}
$$

where $K' = K/C$.

Since $V(0) = A$, we have $V = A\big(1 - K'\,t\big)^2$. Moreover, given that

$$
V(1) = A\big(1 - K'\big)^2 = \frac{1}{4}A
$$

we can solve for $K' = 1/2$, which then gives $V = A\big(1-t/2\big)^2$. The container will be empty when $V = 0$. We see that $A\big(1-t/2\big)^2 = 0$ at $t = 2$, thus it takes 2 minutes to empty the container.

4. Solve the following first-order linear equations, subject to the given conditions.

(b) DE: $x'(t) - \frac{2}{t}x(t) = 1$. SC: $x(1) = 0$.
We begin by finding a multiplying factor

$$m(t) = e^{\int -\frac{2}{t}\,dt} = e^{-2\ln t} = t^{-2}.$$

Multiplying both sides of DE by t^{-2} we have

$$\begin{aligned} t^{-2}x'(t) - 2t^{-3}x(t) &= t^{-2} \\ \frac{d}{dt}\big[t^{-2}x(t)\big] &= t^{-2} \\ t^{-2}x(t) &= \int t^{-2}\,dt + C \\ t^{-2}x(t) &= -t^{-1} + C. \end{aligned}$$

Substituting the side condition $x(1) = 0$ into the last equation gives $0 = -1 + C$, thus $C = 1$ and we find that

$$\boxed{x = t^2(1 - t^{-1}) = t(t-1)}$$

(c) DE: $\sin(x)y'(x) - \cos(x)y(x) = \sin(2x)$. SC: $y(\pi/2) = 0$.
We divide both sides of DE by $\sin(x)$ to arrive at the standard form

$$y'(x) - \cot(x)y(x) = 2\cos(x) \tag{DE$'$}$$

(recall that $\sin(2x) = 2\sin(x)\cos(x)$). Since

$$\int -\cot(x)dx = -\int \frac{\cos(x)}{\sin(x)}\,dx = -\ln\sin(x) = \ln\csc(x),$$

we see that $m(x) = e^{\ln\csc(x)} = \csc(x)$ is a multiplying factor for the standard form (DE$'$). Thus

$$\begin{aligned} \csc(x)y'(x) - \csc(x)\cot(x)y(x) &= 2\frac{\cos(x)}{\sin(x)} \\ \frac{d}{dx}\big[\csc(x)y(x)\big] &= 2\frac{\cos(x)}{\sin(x)} \\ \csc(x)y(x) &= \int 2\frac{\cos(x)}{\sin(x)}\,dx + C \\ \csc(x)y(x) &= 2\ln\big(\sin(x)\big) + C. \end{aligned}$$

Substituting $y(\pi/2) = 0$ into the last equation gives $C = 0$, therefore

$$\boxed{y = 2\sin(x)\ln\big(\sin(x)\big).}$$

(f) DE: $\frac{dy}{dx} = 3y + e^{2x}$. SC: $y(0) = 0$.
The standard form for the differential equation is

$$\text{(DE}') \qquad \frac{dy}{dx} - 3y = e^{2x}$$

and a multiplying factor for (DE′) is $m(x) = e^{-3x}$. Thus

$$\begin{aligned} e^{-3x}\frac{dy}{dx} - 3e^{-3x}y &= e^{-x} \\ \frac{d}{dx}\left[e^{-3x}y\right] &= e^{-x} \\ e^{-3x}y &= \int e^{-x}\,dx + C \\ e^{-3x}y &= -e^{-x} + C. \end{aligned}$$

Substitute the side condition $x = 0$, $y = 0$ to get $0 = C - 1$. Using $C = 1$ and solving the last equation for y gives

$$\boxed{y = e^{3x} - e^{2x}.}$$

6. Find the general solution, $y(x)$, of the following second-order homogeneous linear ODEs.

(b) DE: $y'' - 3y = 0$. AE: $r^2 - 3 = 0$. Roots: $r_1 = \sqrt{3}$, $r_2 = -\sqrt{3}$. Fundamental Solutions: $y_1 = e^{\sqrt{3}x}$, $y_2 = e^{-\sqrt{3}x}$.

$$\boxed{y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}.}$$

(c) DE: $y''+3y = 0$. AE: $r^2+3 = 0$. Roots: $r_1 = i\sqrt{3}$, $r_2 = -i\sqrt{3}$, $\alpha = 0$, $\beta = \sqrt{3}$. Fundamental Solutions: $y_1 = \cos\left(\sqrt{3}x\right)$, $y_2 = \sin\left(\sqrt{3}x\right)$.

$$\boxed{y = c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right).}$$

(e) DE: $y'' - 3y' = 0$. AE: $r^2 - 3r = 0$. Roots: $r_1 = 0$, $r_2 = 3$. Fundamental Solutions: $y_1 = 1$, $y_2 = e^{3x}$.

$$\boxed{y = c_1 + c_2 e^{3x}.}$$

(g) DE: $2y'' + 5y' + 2y = 0$. AE: $2r^2 + 5r + 2 = (2r + 1)(r + 2) = 0$. Roots: $r_1 = -1/2$, $r_2 = -2$. Fundamental Solutions: $y_1 = e^{-x/2}$, $y_2 = e^{-2x}$.

$$\boxed{y = c_1 e^{-x/2} + c_2 e^{-2x}.}$$

(h) DE: $y'' - 6y' + 13y = 0$. AE: $r^2 - 6r + 13 = 0$. Roots: $r_1 = 3 + 2i$, $r_2 = 3 - 2i$, $\alpha = 3$, $\beta = 2$. Fundamental Solutions: $y_1 = e^{3x}\cos(2x)$, $y_2 = e^{3x}\sin(2x)$.

$$\boxed{y = e^{3x}\big(c_1 \cos(2x) + c_2 \sin(2x)\big)}$$

7. Find the particular solutions $y(t)$, meeting the given initial data, of the following second-order homogeneous linear ODEs.

(a) DE: $y'' - 5y' + 6y = 0$.
SC: $y(0) = 1$, $y'(0) = 2$.
AE: $r^2 - 5r + 6 = (r-2)(r-3) = 0$.
Roots: $r_1 = 2$, $r_2 = 3$.
Fundamental Solutions: $y_1 = e^{2t}$, $y_2 = e^{3t}$.
The general solution, $y(t)$, and its first derivative, $y'(t)$, are given by

$$\begin{aligned} y &= c_1 e^{2t} + c_2 e^{3t} \\ y' &= 2c_1 e^{2t} + 3c_2 e^{3t}. \end{aligned}$$

The side conditions SC imply that c_1, c_2 must satisfy the linear equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 + 3c_2 &= 2. \end{aligned}$$

Solving these equations gives $c_1 = 1$, $c_2 = 0$, and thus

$$\boxed{y = e^{2t}.}$$

(b) DE: $y'' - 4y' + 4y = 0$.
SC: $y(0) = 0$, $y'(0) = 1$.
AE: $r^2 - 4r + 4 = (r-2)^2 = 0$.
Roots: $r_1 = 2$, $r_2 = 2$.
Fundamental Solutions: $y_1 = e^{2t}$, $y_2 = te^{2t}$.
The general solution, $y(t)$, and its first derivative, $y'(t)$, are given by

$$\begin{aligned} y &= e^{2t}(c_1 + c_2 t) \\ y' &= e^{2t}\big[2c_1 + c_2(2t+1)\big]. \end{aligned}$$

The side conditions SC imply that c_1, c_2 must satisfy the linear equations

$$\begin{aligned} c_1 &= 0 \\ 2c_1 + c_2 &= 1. \end{aligned}$$

Solving these equations gives $c_1 = 0$, $c_2 = 1$, and thus

$$\boxed{y = te^{2t}.}$$

(e) DE: $5y'' + 8y' + 5y = 0$.
SC: $y(0) = 0$, $y'(0) = 1$.
AE: $5r^2 + 8r + 5 = 0$.
Roots: $r_1 = -\frac{4}{5} + i\frac{3}{5}$, $r_2 = -\frac{4}{5} - i\frac{3}{5}$. $\alpha = -\frac{4}{5}$, $\beta = \frac{3}{5}$.
Fundamental Solutions: $y_1 = e^{-4t/5}\cos(3t/5)$, $y_2 = e^{-4t/5}\sin(3t/5)$.
The general solution, $y(t)$, is given by

$$y = e^{-4t/5}\big(c_1\cos(3t/5) + c_2\sin(3t/5)\big).$$

The side condition $y(0) = 0$ implies that $c_1 = 0$. Given that, the derivative, $y'(t)$, is given by

$$y'(t) = \frac{c_2}{5} e^{-4t/5} \big(3\cos(3t/5) - 4\sin(3t/5)\big).$$

The side condition $y'(0) = 1$ implies that $c_2 = 5/3$, thus

$$\boxed{y = \frac{5}{3} e^{-4t/5} \sin(3t/5).}$$

(f) The DE is the same as in part (e), so we have the same general solution,

$$y = e^{-4t/5} \big(c_1 \cos(3t/5) + c_2 \sin(3t/5)\big).$$

The side condition $y(0) = 1$ implies $c_1 = 1$, thus

$$y = e^{-4t/5} \big(\cos(3t/5) + c_2 \sin(3t/5)\big).$$

The derivative, $y'(t)$, is given by

$$y' = \frac{1}{5} e^{-4t/5} \big[(3c_2 - 4)\cos(3t/5) - (4c_2 + 3)\sin(3t/5)\big].$$

The side condition $y'(0) = 0$ give $c_2 = \frac{4}{3}$, therefore

$$\boxed{y = e^{-4t/5} \big(\cos(3t/5) + \frac{4}{3} \sin(3t/5)\big).}$$

8. DE: $a(x)y'' + b(x)y' + c(x)y = 0$.

(a) Suppose that $y_1(x)$ and $y_2(x)$ are solutions of DE. Let c_1 and c_2 be constants and let $y = c_1y_1 + c_2y_2$ be a superposition of y_1 and y_2, then

$$\begin{aligned} a(x)y'' + b(x)y' + c(x)y &= a(x)[c_1y_1 + c_2y_2]'' \\ &\qquad + b(x)[c_1y_1 + c_2y_2]' + c(x)[c_1y_1 + c_2y_2] \\ &= c_1[a(x)y_1'' + b(x)y_1' + c(x)y_1] \\ &\qquad + c_2[a(x)y_2'' + b(x)y_2' + c(x)y_2] \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

since $a(x)y_1'' + b(x)y_1' + c(x)y_1 = 0$ and $a(x)y_2'' + b(x)y_2' + c(x)y_2 = 0$. We conclude that any superposition of solutions to DE is also a solution to DE.

(b) If $a(x)$, $b(x)$ and $c(x)$ are continuous with $a(x)$ never zero, then the DE in part (a) has a unique solution $y(x)$ with given values $y(x_0)$ and $y'(x_0)$.

Observe that a differentiable function, $y(x)$, has a graph that is tangent to the x-axis at the point x_0 if and only if $y(x_0) = 0$ and $y'(x_0) = 0$. Now observe that the constant function $y(x) = 0$ is a solution to DE and satisfies the conditions $y(x_0) = 0$ and $y'(x_0) = 0$

for any real number x_0. By the uniqueness result stated above, $y(x) = 0$ is the only solution to DE that is tangent to the x-axis at any point x_0.

14. For the differential equation system

$$\text{(DE)} \qquad \begin{aligned} x'(t) &= ax(t) + by(t) \\ y'(t) &= cx(t) + dy(t) \end{aligned}$$

we showed that $x(t)$ must satisfy $x''(t) - (a + d)x'(t) + (ad - bc)x(t) = 0$. Observe that we may rewrite the system DE as follows:

$$\begin{aligned} y'(t) &= dy(t) + cx(t) \\ x'(t) &= by(t) + ax(t). \end{aligned}$$

Applying the previous result we have that

$$\begin{aligned} &y''(t) - (d + a)y'(t) + (da - cb)y(t) \\ &\qquad = y''(t) - (a + d)y'(t) + (ad - bd)y(t) = 0. \end{aligned}$$

Thus $x(t)$ and $y(t)$ satisfy the same second-order differential equation when they are solutions to the first-order system DE.

15. Solve the following system subject to the given initial data

$$\begin{aligned} x'(t) &= x(t) + y(t) & \qquad x(0) &= 1 \\ y'(t) &= -x(t) + y(t) & \qquad y(0) &= 0. \end{aligned}$$

From problem 14 we know that $x(t)$ must be a solution of the second-order DE: $x''(t) - 2x'(t) + 2x(t) = 0$ and satisfy the side conditions SC: $x(0) = 1$, $x'(0) = x(0) + y(0) = 1 + 0 = 1$. The auxiliary equation AE: $r^2 - 2r + 2 = 0$ has non-real roots $\alpha \pm i\beta = 1 \pm i1$. Therefore $x(t) = e^t(c_1 \cos t + c_2 \sin t)$ for some constants c_1, c_2. The side condition $x(0) = 1$ implies that $c_1 = 1$ and $x(t) = e^t(\cos t + c_2 \sin t)$. A direct calculation shows that the derivative of $x(t)$ is $x'(t) = e^t\big[(c_2 + 1)\cos t + (c_2 - 1)\sin t\big]$. The side condition $x'(0) = 1$ implies that $c_2 = 0$. Thus

$$\begin{aligned} x(t) &= e^t \cos t \\ y(t) = x'(t) - x(t) &= e^t\big[\cos t - \sin t\big] - e^t \cos t = -e^t \sin t. \end{aligned}$$

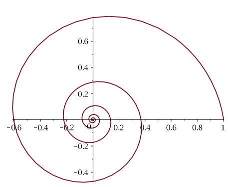

FIGURE 1. Graph of $\big(e^t \cos t, -e^t \sin t\big)$

17. Solve each of the following systems subject to the given initial data:

(a)

$$x' = 3x - 4y \qquad x(0) = 1$$
$$y' = x - y \qquad y(0) = 1.$$

From problem 14 we know that $y(t)$ must be a solution of the second-order DE: $y''(t) - 2y'(t) + y(t) = 0$ and satisfy the side conditions SC: $y(0) = 1$, $y'(0) = x(0) - y(0) = 1 - 1 = 0$. The auxiliary equation AE: $r^2 - 2r + 1 = (r-1)^2 = 0$ has a real root $r = 1$ with multiplicity 2. Therefore $y(t) = (c_1 + c_2 t)e^t$ for some constants c_1, c_2. The side condition $y(0) = 1$ implies that $c_1 = 1$ and $y(t) = (1 + c_2 t)e^t$. A direct calculation shows that the derivative of $y(t)$ is $y'(t) = \big[1 + (1+t)c_2\big]e^t$. The side condition $y'(0) = 0$ implies that $c_2 = -1$. Thus

$$y(t) = (1-t)e^t$$
$$x(t) = y'(t) + y(t) = -te^t + (1-t)e^t = (1-2t)e^t.$$

(b)

$$x' = x - 4y \qquad x(0) = 1$$
$$y' = x + y \qquad y(0) = 1.$$

From problem 14 we know that $x(t)$ must be a solution of the second-order DE: $x''(t) - 2x'(t) + 5x(t) = 0$ and satisfy the side conditions SC: $x(0) = 1$, $x'(0) = x(0) - 4y(0) = 1 - 4 = -3$. The auxiliary equation AE: $r^2 - 2r + 5 = 0$ has non-real roots $\alpha \pm i\beta = 1 \pm i2$. Therefore $x(t) = e^t(c_1 \cos 2t + c_2 \sin 2t)$ for some constants c_1, c_2. The side condition $x(0) = 1$ implies that $c_1 = 1$ and $x(t) = e^t(\cos 2t + c_2 \sin 2t)$. A direct calculation shows that the derivative of $x(t)$ is $x'(t) = e^t\big[(1+2c_2)\cos 2t + (c_2 - 2)\sin 2t\big]$. The side condition $x'(0) = -3$ implies that $c_2 = -2$. Thus

$$x(t) = e^t(\cos 2t - 2\sin 2t)$$
$$y(t) = \frac{1}{4}(x(t) - x'(t)) = e^t(\cos 2t + \frac{1}{2}\sin 2t).$$

(c)

$$x' = x + 2y \qquad x(0) = 0$$
$$y' = 3x + 4y \qquad y(0) = 1.$$

From problem 14 we know that $x(t)$ must be a solution of the second-order DE: $x''(t) - 5x'(t) - 2x(t) = 0$ and satisfy the side conditions SC: $x(0) = 0$, $x'(0) = x(0) + 2y(0) = 0 + 2 = 2$. The auxiliary equation AE: $r^2 - 5r - 2 = 0$ has real roots $r_1 = (5 + \sqrt{33})/2$, $r_2 = (5 - \sqrt{33})/2$. Therefore $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for some constants c_1, c_2. The side condition $x(0) = 0$ implies that $c_1 + c_2 = 0$ and $x(t) = c_1(e^{r_1 t} - e^{r_2 t})$. A direct calculation shows that the derivative of $x(t)$ is $x'(t) = c_1(r_1e^{r_1t} - r_2e^{r_2t})$. The side condition $x'(0) = 2$ implies that $c_1 = 2/(r_1 - r_2) = 2/\sqrt{33}$. Thus

$$\begin{aligned} x(t) &= 2(e^{r_1t} - e^{r_2t})/\sqrt{33} \\ &= 2\big(e^{(5+\sqrt{33})t/2} - e^{(5-\sqrt{33})t/2}\big)/\sqrt{33}, \\ y(t) &= (x'(t) - x(t))/2 \\ &= \big[c_1(r_1e^{r_1t} - r_2e^{r_2t}) - c_1(e^{r_1t} - e^{r_2t})\big]/2 \\ &= \big[(r_1 - 1)e^{r_1t} - (r_2 - 1)e^{r_2t}\big]c_1/2 \\ &= \big[(3+\sqrt{33})e^{(5+\sqrt{33})t/2} - (3-\sqrt{33})e^{(5-\sqrt{33})t/2}\big]/(2\sqrt{33}). \end{aligned}$$

18. For any complex number z, we define the hyperbolic sine and cosine by

$$\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right), \quad \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right).$$

(a) Adding and subtracting the above gives $e^z = \cosh(z) + \sinh(z)$ and $e^{-z} = \cosh(z) - \sinh(z)$. Therefore

$$\begin{aligned} \cosh^2(z) - \sinh^2(z) &= \big(\cosh(z) + \sinh(z)\big)\big(\cosh(z) - \sinh(z)\big) \\ &= e^z e^{-z} \\ &= 1. \end{aligned}$$

(b)

$$\begin{aligned} \frac{d}{dx}\sinh x &= \frac{d}{dx}\left[\frac{1}{2}\left(e^x - e^{-x}\right)\right] = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x, \\ \frac{d}{dx}\cosh x &= \frac{d}{dx}\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right] = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x. \end{aligned}$$

(c) Recall that $e^{iy} = \cos y + i\sin y$, $e^{-iy} = \cos y - i\sin y$. Therefore

$$\begin{aligned} \cosh(iy) &= \frac{1}{2}\left(e^{iy} + e^{-iy}\right) = \cos y, \\ \sinh(iy) &= \frac{1}{2}\left(e^{iy} - e^{-iy}\right) = i\sin y. \end{aligned}$$

(d) Define

$$\begin{aligned} \cos z &= \cosh(iz) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ \sin z &= \frac{1}{i}\sinh(iz) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right). \end{aligned}$$

Then

$$\begin{aligned} \cos^2 z + \sin^2 z &= \Big[\cosh(iz)\Big]^2 + \Big[\frac{1}{i}\sinh(iz)\Big]^2 \\ &= \cosh^2(iz) - \sinh^2(iz) \\ &= 1 \end{aligned}$$

by part (a). Thus $\cos^2 z + \sin^2 z = 1$.

20. Show that the general solution of a second-order homogeneous linear equation

$$\text{(DE)} \qquad ay'' + by' + cy = 0$$

($a \neq 0$, b and c are constants) is of the form $c_1 y_1(x) + c_2 y_2(x)$, where $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions (i.e., neither is a constant multiple of the other).

(a) Suppose $g(x)/f(x)$ is differentiable on an open interval I. Then g and f are linearly dependent if and only if $g(x)/f(x)$ is a constant, and this can happen if and only if

$$\frac{d}{dx}\left[\frac{g(x)}{f(x)}\right] = \frac{f(x)g'(x) - f'(x)g(x)}{[f(x)]^2} = \frac{W[f,g](x)}{[f(x)]^2} = 0.$$

Thus g and f are linearly dependent if and only if $W[f,g](x) = 0$ for all x in I.

If $g(x)/f(x)$ is not differentiable on I but $f(x)/g(x)$ is differentiable on I, then we conclude that g and f are linearly dependent if and only if $W[g,f](x) = 0$ for every $x \in I$. But

$$W[g,f](x) = -W[f,g](x)$$

so the original condition $W[f,g](x) = 0$ still applies.

(b) Let $y(x)$ and $z(x)$ be any two solutions of (DE). Then

$$\begin{aligned} \frac{d}{dx}\big[W[y,z](x)\big] &= \frac{d}{dx}\big[yz' - y'z\big] \\ &= (y'z' + yz'') - (y'z' + y''z) \\ &= yz'' - y''z. \end{aligned}$$

Multiplying by the non-zero constant a gives

$$\begin{aligned} a\frac{d}{dx}\big[W[y,z](x)\big] &= y(az'') - (ay'')z \\ &= y(-bz' - cz) - (-by' - cy)z \\ &= -b(yz' - y'z) \\ &= -bW[y,z](x) \end{aligned}$$

(we used the fact that y or z are solutions of (DE) to substitute $ay'' = -by' - cy$ and $az'' = -bz' - cz$). Thus $W[y,z](x)$ satisfies the differential equation

$$aW'(x) + bW(x) = 0,$$

and therefore $W[y,z](x) = Ce^{-bx/a}$ for some constant C.

(c) Since $e^{-bx/a}$ is never zero, $W[y,z](x) = 0$ if and only if $C = 0$, in which case $W[y,z](x) = 0$ for all x.

(d) Let $z = d_1y_1 + d_2y_2$, where d_1 and d_2 are constants. Then

$$\begin{aligned} W[y,z] &= W[y, d_1y_1 + d_2y_2] \\ &= y(d_1y_1 + d_2y_2)' - y'(d_1y_1 + d_2y_2) \\ &= y(d_1y_1' + d_2y_2') - y'(d_1y_1 + d_2y_2) \\ &= d_1(yy_1' - y'y_1) + d_2(yy_2' - y'y_2) \\ &= d_1W[y,y_1] + d_2W[y,y_2]. \end{aligned}$$

Fix an x_0 and let $d_1 = -W[y,y_2](x_0)$, $d_2 = W[y,y_1](x_0)$. Then

$$\begin{aligned} W[y,z](x_0) = &-W[y,y_2](x_0)W[y,y_1](x_0) \\ &+ W[y,y_1](x_0)W[y,y_2](x_0) = 0. \end{aligned}$$

So we can choose the constants d_1 and d_2 so that $W[y,z](x) = 0$ at one point, $x = x_0$.

(e) From part (c) we may conclude that since $W[y,z]$ vanishes at one point, $W[y,z](x) = 0$ for all x.

(f) From part (a), y must be a constant multiple of z, i.e.

$$\begin{aligned} y &= c'z \\ &= c'(d_1y_1 + d_2y_2) \\ &= c_1y_1 + c_2y_2, \end{aligned}$$

where $c_1 = c'd_1$, $c_2 = c'd_2$. Thus every solution is a linear combination of two independent solutions.