Proceedings of Symposia in Pure Mathematics
Volume **64**, 1999

Centro-affine Geometry in the Plane and Feedback Invariants of Two-state Scalar Control Systems

George R. Wilkens

Abstract. The goal of this paper is to establish the precise connection between the centro-affine invariants of plane curves and the feedback invariants of nonlinear scalar control systems in the plane. We will also show how the centro-affine structure provides a shortcut to the structure equations for feedback equivalence one obtains by applying Cartan's equivalence method.

0. Introduction

A direct observation of the way that control systems transform under the action of feedback transformations leads us to consider a classical problem in the geometry of submanifolds of $\mathbf{R}^n$. Specifically, let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$ represent a control system, where $\mathbf{x} \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$, with $m < n$. Let $\mathbf{x}$ be any point in state space and let $T_{\mathbf{x}}$ denote the tangent space at $\mathbf{x}$. The control system determines the mapping $u \mapsto \mathbf{f}(\mathbf{x}, u)$ from the control space, $\mathbf{R}^m$, to the tangent space $T_{\mathbf{x}}$. If we make the reasonable assumption that the rank of $\partial\mathbf{f}/\partial u$ is maximal, then we see that this mapping determines an m-dimensional parameterized submanifold of $T_{\mathbf{x}}$. As we vary $\mathbf{x}$, the control system $\mathbf{f}(\mathbf{x}, u)$ determines an n-parameter family of submanifolds, and we are interested in the action of feedback transformations on this family.

Let $\bar{\mathbf{x}} = \boldsymbol{\phi}(\mathbf{x})$, $\bar{u} = \psi(\mathbf{x}, u)$ be a feedback transformation taking $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$ to $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}(\bar{\mathbf{x}}, \bar{u})$. To see how the submanifolds $\bar{u} \mapsto \bar{\mathbf{f}}(\bar{\mathbf{x}}, \bar{u})$ are related to the original submanifolds, we need only apply the chain rule. Since

$$\bar{\mathbf{f}} = \frac{d\bar{\mathbf{x}}}{dt} = \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = \frac{\partial\boldsymbol{\phi}}{\partial\mathbf{x}} \cdot \mathbf{f},$$

we see that applying the nonsingular matrix $\partial\boldsymbol{\phi}/\partial\mathbf{x}$ to the image of $u \mapsto \mathbf{f}(\mathbf{x}, u)$ yields the image of $\bar{u} \mapsto \bar{\mathbf{f}}(\bar{\mathbf{x}}, \bar{u})$; moreover we have also reparametrized the image with the new parameter $\bar{u} = \psi(\mathbf{x}, u)$. Since $\boldsymbol{\phi}(\mathbf{x})$ can be any diffeomorphism, $\partial\boldsymbol{\phi}/\partial\mathbf{x}$ can be any GL(n,$\mathbf{R}$) matrix at $\mathbf{x}$. Together, these observations show that

1991 *Mathematics Subject Classification.* 49,53.

Key words and phrases. Feedback equivalence, control systems, centro-affine geometry, equivalence method, moving frames.

Supported by the Ministry of Colleges and Universities of Ontario and the Natural Sciences and Engineering Research Council of Canada

the feedback transformations induce an arbitrary reparametrization and an arbitrary general linear action on each submanifold $u \mapsto \mathbf{f}(\mathbf{x}, u)$, which introduces the problem of classifying the general linear orbits of submanifolds of $\mathbf{R}^n$.

For this classical submanifold problem, we view GL(n,$\mathbf{R}$) as a group of motions on $\mathbf{R}^n$ and try to classify submanifolds of $\mathbf{R}^n$ up to a GL(n,$\mathbf{R}$)-motion. This problem is exactly analogous to the problem of classifying submanifolds of $\mathbf{R}^n$ up to a motion of the Euclidean group in $\mathbf{R}^n$. The term *centro-affine geometry* is used to describe the geometry on $\mathbf{R}^n$ induced by GL(n,$\mathbf{R}$), just as Euclidean geometry is used to describe the geometry on $\mathbf{R}^n$ induced by the Euclidean group. Once we know the centro-affine invariants of submanifolds of $\mathbf{R}^n$, our expectation is that these invariants should also appear in some form as feedback invariants of a control system. When we know how to recognize them, we will be able to assign geometric meaning to them and use our centro-affine insight to interpret them. We will show exactly how this occurs in the special case of a scalar control system in the plane.

In n-dimensional Euclidean geometry, we often make use of frame fields to study submanifolds. For example, the Frenet frame on a regular curve is a Euclidean invariant of the curve, and this frame determines the arc-length and $n-1$ curvatures of the curve. These functions are also Euclidean invariants, and they classify the curve up to a Euclidean motion. In an entirely analogous manner, we can associate a GL(n,$\mathbf{R}$)-invariant frame field to a curve, and this frame field determines an "arc-length" and $n-1$ "curvatures" which classify the curve up to GL(n,$\mathbf{R}$)-motions [**GW97**].

In section 1 we use Cartan's method of moving frames [**Ca51**] to construct a centro-affine invariant frame field for a curve in the plane. For a regular class of curves we are able to find a GL(2,$\mathbf{R}$)-invariant curve parameter, which we call the centro-affine arc-length, and an invariant function we call the centro-affine curvature. The arc-length and curvature determine the curve up to a centro-affine motion. In section 2, we show how a general nonlinear feedback transformation induces centro-affine actions on a family of curves naturally associated to a scalar control system in the plane. Then we show exactly how the centro-affine invariants of plane curves appear as feedback invariants of the control system. In effect, we solve the problem of finding a complete set of invariants for feedback equivalence. Finally, in section 3, we consider some examples and exhibit some normal forms.

1. Centro-affine geometry in the punctured plane.

Moving frames on the punctured plane. The method of moving frames is a technique that is well suited to the study of submanifolds of a homogeneous space. In [**Ca51**], Cartan shows that when a Lie group acts transitively and effectively on a manifold, one can construct a bundle of frames over the manifold. Cartan develops this theory to study two submanifold problems: the problem of contact and the problem of equivalence. The first problem involves determining the order of contact two submanifolds have at a point. The second problem involves determining when there exists an element of the given Lie group that translates one submanifold onto another. Cartan devotes most of his book to specific geometric examples of these problems. More recent descriptions of the method of moving frames can be found in [**Gn78**], [**Gr74**] and an excellent recent book [**Sh96**]. Green's paper [**Gn78**] and Sharpe's book [**Sh96**] carefully point out subtle features in the theory and include many examples. Our goal in this section will be to classify planar curves up to general linear actions.

We define the *punctured plane* to be the plane of column vectors, $\mathbf{R}^2$, with the origin removed; and we will denote it by $\mathbf{R}^2_0$. Since the group GL(2,$\mathbf{R}$) acts transitively and effectively on $\mathbf{R}^2_0$, we can use the method of moving frames to study the GL(2,$\mathbf{R}$)-invariants of curves in the punctured plane. A frame on $\mathbf{R}^2_0$ will consist of a pair of linearly independent vectors in $\mathbf{R}^2_0$, $(\mathbf{e}_0, \mathbf{e}_1)$. Let $\mathcal{F}$ be the set of all frames on $\mathbf{R}^2_0$. We may also think of the frame $(\mathbf{e}_0, \mathbf{e}_1)$ as the GL(2,$\mathbf{R}$) matrix $({e^i}_j)_{0\le i,j\le 1}$, where $\mathbf{e}_j$ is the column vector $\mathbf{e}_j = {}^t({e^0}_j, {e^1}_j)$ (we use ^{t} on the left to denote transpose). Now GL(2,$\mathbf{R}$) is an open subset of the four dimensional vector space L(2,$\mathbf{R}$) of all two by two matrices. The projections

$$
\begin{array}{ccc}
 & \mathrm{L}(2,\mathbf{R}) & \\
 {\scriptstyle \mathbf{e}_0}\swarrow & & \searrow{\scriptstyle \mathbf{e}_1} \\
\mathbf{R}^2 & & \mathbf{R}^2
\end{array}
$$

from L(2,$\mathbf{R}$) onto the left and right columns of the matrices are linear maps, and therefore their restrictions to any open subset are differentiable. In particular, we have two differentiable mappings $\mathbf{e}_0, \mathbf{e}_1 \colon \mathcal{F} \to \mathbf{R}^2_0$. Expressing the derivatives of these mappings in terms of themselves gives the structure equations on $\mathcal{F}$:

$$
\begin{aligned}
d\mathbf{e}_0 &= \mathbf{e}_0\, {\omega^0}_0 + \mathbf{e}_1\, {\omega^1}_0 \\
d\mathbf{e}_1 &= \mathbf{e}_0\, {\omega^0}_1 + \mathbf{e}_1\, {\omega^1}_1 \\
d{\omega^i}_j &= -{\omega^i}_0 \wedge {\omega^0}_j - {\omega^i}_1 \wedge {\omega^1}_j \qquad 0 \le i, j \le 1.
\end{aligned} \tag{1}
$$

In these equations we view $\mathbf{e}_0$ and $\mathbf{e}_1$ as functions from $\mathcal{F}$ to $\mathbf{R}^2$. The equations for $d{\omega^i}_j$ are consequences of $d^2\mathbf{e}_0 = d^2\mathbf{e}_1 = 0$. The 1-forms ${\omega^0}_0, {\omega^0}_1, {\omega^1}_0, {\omega^1}_1$ form a basis for the 1-forms on $\mathcal{F}$.

Zeroth order frames. Let $I \subset \mathbf{R}$ be an open interval and let $\mathbf{x}\colon I \to \mathbf{R}^2_0$ be a smooth immersed curve. We call

$$\{(u; \mathbf{e}_0, \mathbf{e}_1) \in I \times \mathcal{F} \mid \mathbf{e}_0 = \mathbf{x}(u)\}$$

the set of *zeroth order frames* for the curve $\mathbf{x}$, and we denote this set by $\mathcal{F}^{(0)}_{\mathbf{x}}$. We think of the zeroth order frames as the restriction of the frames on $\mathbf{R}^2_0$ to the curve $\mathbf{x}$ by requiring that the first leg of the frame, $\mathbf{e}_0$, be the position vector of the curve; that is we let $\mathbf{e}_0(u) = \mathbf{x}(u)$ for all $u \in I$. If we differentiate this equation and use the structure equations (1), we get that

$$\mathbf{x}'(u)\, du = d\mathbf{x} = d\mathbf{e}_0 = \mathbf{e}_0\, {\omega^0}_0 + \mathbf{e}_1\, {\omega^1}_0. \tag{2}$$

This implies that ${\omega^0}_0$ and ${\omega^1}_0$ are multiples of du. In Cartan's language, the curve parameter u is called a *principal parameter*, and the components of $d\mathbf{e}_j$ that are multiples of du are called *principal components*. We want to choose the second leg of the frame, $\mathbf{e}_1$, in a way that is adapted to the geometry of the curve. The natural thing to do is to choose $\mathbf{e}_1$ to be tangent to the curve, however this is not always possible. Since $\mathbf{e}_0$ and $\mathbf{e}_1$ must be linearly independent, we can require $\mathbf{e}_1$ to be tangent to the curve if and only if $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly independent. Since $\mathbf{x}(u) \wedge \mathbf{x}'(u)\, du = \mathbf{e}_0 \wedge d\mathbf{e}_0 = \mathbf{e}_0 \wedge \mathbf{e}_1\, {\omega^1}_0$, we see that $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly independent if and only if ${\omega^1}_0 \neq 0$. Since $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly dependent if and only if the curve is tangent to the ray through $\mathbf{x}(u)$ emanating from the origin, we see that ${\omega^1}_0$ vanishes exactly when the curve is tangent to radial directions. We will consider the two extreme cases for this condition: ${\omega^1}_0$ either vanishes or is nonzero for every point on the curve. We will encounter several similar conditions in the rest of this paper. We will use the term *regular* to mean that a condition is satisfied for every point in the domain, and we will always restrict our consideration to regular cases. If we consider the regular case satisfying ${\omega^1}_0 = 0$ for every point on the curve, then $d\mathbf{e}_0 \wedge \mathbf{e}_0 = 0$ for all $u \in I$. This implies that the ray through $\mathbf{e}_0$ is constant, and we see that the curve $\mathbf{x}(u)$ lies on a ray.

First order frames. We now restrict to the regular case where ${\omega^1}_0 \neq 0$ for all $u \in I$. In this case we can define the first order frames $\mathcal{F}^{(1)}_{\mathbf{x}}$ to be the set

$$\mathcal{F}^{(1)}_{\mathbf{x}} = \{(u; \mathbf{e}_0, \mathbf{e}_1) \in \mathcal{F}^{(0)}_{\mathbf{x}} \mid \mathbf{e}_1 \text{ points in the direction of } \mathbf{x}'(u)\}.$$

Remark. This choice of first order frames is adapted to *oriented* curves. If we allow the curve to reverse orientation, then we can only require that $\mathbf{e}_1$ be tangent to the curve, but not that it point in a particular direction.

Since $\mathbf{x}'$ is parallel to $\mathbf{e}_1$, equation (2) implies that $d\mathbf{e}_0 = \mathbf{e}_1\, {\omega^1}_0$; equivalently ${\omega^0}_0 = 0$. This shows that ${\omega^1}_0$ is a non-zero multiple of du, so a 1-form on $\mathcal{F}^{(1)}_{\mathbf{x}}$ is a multiple of du if and only if it is a multiple of ${\omega^1}_0$. In particular, principal components must be multiples of ${\omega^1}_0$. Differentiating the last relation and using equations (1) gives $0 = d{\omega^0}_0 = -{\omega^0}_1 \wedge {\omega^1}_0$. Therefore ${\omega^0}_1$ is a principal component and we may write ${\omega^0}_1 = a\, {\omega^1}_0$, for some function a. A subgroup of GL(2,$\mathbf{R}$) acts on the function a, and we can compute the infinitesimal action by differentiating both sides of the last equation. (See pages 40–43 of [**Ga89**] for a discussion of group actions and how to identify them from their infinitesimal actions.) Differentiating both sides, we get

$$\begin{aligned} d{\omega^0}_1 &= -{\omega^0}_1 \wedge {\omega^1}_1 \\ &= -a\, {\omega^1}_0 \wedge {\omega^1}_1 \qquad \text{and} \\ d(a\, {\omega^1}_0) &= da \wedge {\omega^1}_0 - a\, {\omega^1}_1 \wedge {\omega^1}_0. \end{aligned}$$

Subtracting the second equation from the third equation we arrive at the equation $0 = (da - 2a\, {\omega^1}_1) \wedge {\omega^1}_0$. This shows that the function a is acted on by a square. We can in fact compute the function a explicitly. The first order frames are parametrized by $I \times \mathbf{R}_+$, where (u, ν) determines the frame $(\mathbf{e}_0, \mathbf{e}_1) = (\mathbf{x}(u), \mathbf{x}'(u)\, \nu^{-1})$, $\nu > 0$. Hence, $d\mathbf{e}_0 = \mathbf{x}'(u)\, du = (\mathbf{x}'(u)\, \nu^{-1})(\nu\, du) = \mathbf{e}_1\, \nu\, du = \mathbf{e}_1\, {\omega^1}_0$, and we see that ${\omega^1}_0 = \nu\, du$. By taking determinants we can isolate ${\omega^0}_1$ in the structure equation for $d\mathbf{e}_1$. Explicitly,

$$\det(d\mathbf{e}_1, \mathbf{e}_1) \overset{\text{by }(1)}{=} \det(\mathbf{e}_0, \mathbf{e}_1)\, {\omega^0}_1 = \det(\mathbf{x}(u), \mathbf{x}'(u))\, \nu^{-1}\, {\omega^0}_1.$$

We also have that

$$\det(d\mathbf{e}_1, \mathbf{e}_1) = \det(d(\mathbf{x}'(u)\, \nu^{-1}), \mathbf{x}'(u)\, \nu^{-1}) = \det(\mathbf{x}''(u), \mathbf{x}'(u))\, \nu^{-2}\, du,$$

so

$${\omega^0}_1 = \frac{\det(\mathbf{x}''(u), \mathbf{x}'(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u))}\, \nu^{-1}\, du = \frac{\det(\mathbf{x}''(u), \mathbf{x}'(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u))}\, \nu^{-2}\, {\omega^1}_0 = a\, {\omega^1}_0.$$

We see that the function $a = [\det(\mathbf{x}''(u), \mathbf{x}'(u)) / \det(\mathbf{x}(u), \mathbf{x}'(u))]\, \nu^{-2}$, and the square action of ν^{-1} is evident. From this formula it is clear that a vanishes if and only if $\mathbf{x}''(u)$ and $\mathbf{x}'(u)$ are linearly dependent, which occurs exactly at the inflection points of the curve. If we consider the regular case where a is identically zero, we see that this happens if and only if every point on the curve is an inflection point. This means that the curve must lie on a line, and since ${\omega^1}_0 \neq 0$, this line can not pass through the origin. Thus a identically zero classifies the affine lines.

REMARK. We do not need the explicit formula for a to see that the vanishing of a determines affine lines. This is because the vanishing of a is equivalent to the condition $d\mathbf{e}_1 \wedge \mathbf{e}_1 = 0$. Since $\mathbf{e}_1$ determines the direction of the tangent line, this equation says that the tangent line is constant and therefore $\mathbf{x}(u)$ lies on a line.

Since a is acted on by a square, the sign of a is an invariant. The formula for a makes it clear that a will be positive when the position vector $\mathbf{x}(u)$ and the acceleration vector $\mathbf{x}''(u)$ point to the same side of the tangent line, and a will be negative when the acceleration vector and the position vector point to opposite sides of the tangent line. In other words, that a is negative when the curve is convex, and a is positive when the curve is concave.

Second order frames. We now consider the last regular case, namely the case where $a \neq 0$ for every point on the curve. In this case we can determine a unique set of second order frames, $\mathcal{F}^{(2)}_{\mathbf{x}}$, by choosing the first order frames for which $a = -\epsilon$, where $\epsilon = 1$ if the curve is convex and $\epsilon = -1$ if the curve is concave. With $a = -\epsilon$, the parameter ν takes on the value $\nu = \sqrt{|\det(\mathbf{x}''(u), \mathbf{x}'(u))/\det(\mathbf{x}(u), \mathbf{x}'(u))|}$, and the frame $(\mathbf{e}_0, \mathbf{e}_1) = (\mathbf{x}(u), \mathbf{x}'(u)\,\nu^{-1})$ is uniquely defined along the curve. The 1-form ${\omega^1}_0 = \nu\, du = \sqrt{|\det(\mathbf{x}''(u), \mathbf{x}'(u))/\det(\mathbf{x}(u), \mathbf{x}'(u))|}\, du$ is well defined on the curve, and determines the *centro-affine arc-length* of the curve. We will denote the centro-affine arc-length parameter by s, which implies that ${\omega^1}_0 = ds$. Notice that the centro-affine arc-length is second order in $\mathbf{x}(u)$, as is the vector $\mathbf{e}_1$. The 1-form ${\omega^1}_1$, which is obtained by differentiating $\mathbf{e}_1$, must be a multiple of ds. We will write ${\omega^1}_1 = \kappa(s)\, ds$, and we will call κ the *centro-affine curvature* of the curve. Notice that κ is third order.

The structure equations (1) reduce to the Frenet type equations

$$\begin{aligned} d\mathbf{e}_0 &= \qquad\qquad \mathbf{e}_1\, ds \\ d\mathbf{e}_1 &= -\epsilon\, \mathbf{e}_0\, ds + \mathbf{e}_1\, \kappa\, ds. \end{aligned} \tag{3}$$

A standard argument shows that the centro-affine arc-length and the centro-affine curvature determine the curve $\mathbf{x}(u)$ up to a GL(2,$\mathbf{R}$)-motion. It is also easy to check that if you reverse the orientation of the curve, then κ changes sign. Thus the centro-affine arc-length and the absolute value of κ determine unoriented curves up to a GL(2,$\mathbf{R}$)-motion.

Structure theory of centro-affine curves. We can collect the previous results into a classification theorem for centro-affine curves. Essentially the same results appear in the Shirokows' book on affine geometry [**SS62,pp. 49–51, pp. 57–61**]. They attribute these results to Mayer and Myller [**MM33**].

THEOREM (MAYER AND MYLLER). *Let $I \subset \mathbf{R}$ be an open interval and let $\mathbf{x}\colon I \to \mathbf{R}^2_0$ be a smooth immersed curve. We have the following three regular classes of curves.*

- *If $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly dependent for all u in I, then the curve lies on a ray emanating from the origin.*

- *If* $\mathbf{x}(u)$ *and* $\mathbf{x}'(u)$ *are linearly independent and* $\mathbf{x}'(u)$ *and* $\mathbf{x}''(u)$ *are linearly dependent for all* u *in* I*, then the curve lies on an affine line.*
- *If* $\mathbf{x}(u)$ *and* $\mathbf{x}'(u)$ *are linearly independent and* $\mathbf{x}'(u)$ *and* $\mathbf{x}''(u)$ *are linearly independent for all* u *in* I*, then the curve possesses a GL(2,*$\mathbf{R}$*)-invariant arc-length* s *and a GL(2,*$\mathbf{R}$*)-invariant frame* $(\mathbf{e}_0(s), \mathbf{e}_1(s))$ *satisfying structure equations (3). The arc-length* s *and the curvature* $\kappa(s)$ *determine the curve up to a GL(2,*$\mathbf{R}$*)-motion.*

The following observation provides some insight to the meaning of the arc-length parameter s. If the position vector and the velocity vector of the curve are linearly independent, then we can use $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ as a basis for $\mathbf{R}^2_0$. Therefore, we can write the acceleration vector in terms of this basis, giving

$$\mathbf{x}''(u) = \alpha(u)\,\mathbf{x}(u) + \beta(u)\,\mathbf{x}'(u).$$

If we reparametrize the curve, letting $u = u(t)$, then we get

$$\mathbf{x}''(t) = \hat{\alpha}(t)\,\mathbf{x}(t) + \hat{\beta}(t)\,\mathbf{x}'(t), \quad \text{where } \hat{\alpha}(t) = u'(t)^2\,\alpha(u(t)).$$

We can always reparametrize the curve so that $\hat{\alpha}(t) = -\epsilon$, $\epsilon = \pm 1$. With this parametrization we have $\mathbf{x}''(t) = -\epsilon\,\mathbf{x}(t) + \hat{\beta}(t)\,\mathbf{x}'(t)$. If we write this as a system of equation, $\mathbf{x}'(t) = \mathbf{e}_1(t)$, $\mathbf{e}_1'(t) = -\epsilon\,\mathbf{x}(t) + \hat{\beta}(t)\,\mathbf{e}_1(t)$, we see that this system exactly matches the structure equations (3) (recall that $\mathbf{e}_0$ is the position vector, $\mathbf{x}$). Thus t must be the arc-length parameter s and $\hat{\beta}$ must be the curvature κ. We see that the arc-length parameter is the one which normalizes the coefficient of $\mathbf{x}$ when we express the acceleration vector in terms of the position vector and the velocity vector. The coefficient of $\mathbf{x}'$ is then the curvature. Another interpretation of the arc-length parameter is that it is the parameter for which $\det(\mathbf{x}''(s), \mathbf{x}'(s)) = -\epsilon\ \det(\mathbf{x}(s), \mathbf{x}'(s))$, since in a general parameter we would have $\det(\mathbf{x}''(u), \mathbf{x}'(u)) = \alpha(u)\ \det(\mathbf{x}(u), \mathbf{x}'(u))$.

The previous paragraph gives us one interpretation of the curvature κ, it is the coefficient of the velocity vector when we express the acceleration in terms of the position vector and the velocity vector (all derivatives are with respect to arc-length). However, this is not a very satisfying description of the curvature. To get a better description of the curvature, notice that the frame $(\mathbf{e}_0(s), \mathbf{e}_1(s))$ determines the volume element $\mathbf{e}_0 \wedge \mathbf{e}_1$ at each point of the curve. If we differentiate this volume element with respect to arc-length and use the structure equations (3), we get

$$d(\mathbf{e}_0 \wedge \mathbf{e}_1) = d\mathbf{e}_0 \wedge \mathbf{e}_1 + \mathbf{e}_0 \wedge d\mathbf{e}_1 = \mathbf{e}_0 \wedge \mathbf{e}_1\,\kappa(s)\,ds.$$

We see that the curvature is the relative rate of change of the volume element $\mathbf{e}_0 \wedge \mathbf{e}_1$ with respect to arc-length. In terms of determinants, this is equivalent to $\kappa(s) = d[\ln|\det(\mathbf{x}(s), \mathbf{x}'(s))|]/ds$. Those curves for which κ is identically zero satisfy Kepler's second law of sweeping out equal area in equal time, although for centro-affine curves time is replaced by arc-length.

We can give a good description of the curves with constant centro-affine curvature. Structure equations (3) imply that a centro-affine curve must satisfy the second order differential equation $\mathbf{x}''(s) = \kappa\,\mathbf{x}'(s) - \epsilon\,\mathbf{x}(s)$. The characteristic equation for this differential equation is $r^2 - \kappa\,r + \epsilon = 0$. The two roots λ_1 and λ_2 are such that $\lambda_1 + \lambda_2 = \kappa$ and $\lambda_1\,\lambda_2 = \epsilon$.

In the concave case, $\epsilon = -1$ which means that the roots have opposite sign. Let λ_1 be the positive root. There are two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $\mathbf{x}(s) =$

$\mathbf{w}_1 e^{\lambda_1 s} + \mathbf{w}_2 e^{\lambda_2 s}$. Since $\mathbf{x}(0)$ and $\mathbf{x}'(0)$ must be linearly independent, $\mathbf{w}_1$ and $\mathbf{w}_2$ must also be linearly independent. Thus, we can apply a GL(2,$\mathbf{R}$) matrix to the curve and put it in the standard form $\mathbf{x}(s) = {}^t(1,0)\, e^{\lambda_1 s} + {}^t(0,1)\, e^{\lambda_2 s}$. We see that the curve is a saddle solution to a linear ordinary differential equation, and up to a general linear action lies on the curve $y^{\lambda_1} = x^{\lambda_2}$, $\lambda_1 \lambda_2 = -1$, $\lambda_1 + \lambda_2 = \kappa$.

The convex case branches into three subcases depending on the sign of $\kappa^2 - 4$. If $|\kappa| > 2$, then we have distinct real roots λ_1 and λ_2 and the solution curve can be put in the standard form $\mathbf{x}(s) = {}^t(1,0)\, e^{\lambda_1 s} + {}^t(0,1)\, e^{\lambda_2 s}$, $\lambda_1 \lambda_2 = 1$, $\lambda_1 + \lambda_2 = \kappa$. We see that the curve is a nodal solution to a linear ordinary differential equation, and up to a general linear action lies on the curve $y^{\lambda_1} = x^{\lambda_2}$.

If $|\kappa| = 2$, then we have two equal real roots λ. Since $2\lambda = \kappa$, λ is either 1 or -1. The solution curve can be put in the standard form $\mathbf{x}(s) = {}^t(1,0)\, e^{\lambda s} + {}^t(0,1)\, s\, e^{\lambda s}$. We see that the curve is an improper node, and lies on the curve $\lambda\, y = x \ln(x)$. If $\kappa = 2$, then $\lambda = 1$ and $\mathbf{x}(s)$ moves away from the origin, while if $\kappa = -2$, then $\lambda = -1$ and $\mathbf{x}(s)$ moves toward from the origin.

If $|\kappa| < 2$, then we have a non-real conjugate pair of roots, λ and $\bar{\lambda}$, with κ being two times the real part of λ. If we let λ be the root with positive imaginary part, then $\lambda = \kappa/2 + \sqrt{-1}\, b$, where b is the positive root of the equation $\kappa^2/4 + b^2 = 1$. We can put the solution curve into the standard form $\mathbf{x}(s) = e^{\kappa s/2}[{}^t(1,0)\cos(b\, s) + {}^t(0,1)\sin(b\, s)]$. If $|\kappa| > 0$ we have a spiral solution which moves away from the origin if κ is positive and moves towards the origin if κ is negative. When $\kappa = 0$ we have a circle centered at the origin.

Notice that when $\kappa = 0$, our curve is a hyperbola in the concave case and a circle in the convex case. Thus $\kappa = 0$ exactly picks out the non-degenerate quadric curves centered at the origin. This fits in with the centro-affine theory of hypersurfaces. In this lowest dimensional case, the symmetric form ${\omega^0}_1 {\omega^1}_0 = -\epsilon\, (ds)^2$ is the Blaschke form, and $\kappa\, (ds)^3$ is the Pick form [**Bl23**]. It is exactly the vanishing of the Pick form that characterizes the non-degenerate quadric hypersurfaces.

2. Two state scalar control

Our goal in this section is to use our new intuition for centro-affine curves to compute the feedback invariants of a two state, single input control system. We will also have the benefit of a geometric interpretation of these invariants. The motivation for considering centro-affine geometry comes from viewing a control system as a submanifold of the tangent bundle, and observing the effect that feedback has on this submanifold.

Consider the control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u), \tag{4}$$

where $\mathbf{x} = {}^t(x^1, x^2) \in \mathbf{R}^2$, $u \in \mathbf{R}$ and $\mathbf{f}(x^1, x^2, u)$ is a smooth function from $\mathbf{R}^2 \times \mathbf{R}$ to $\mathbf{R}^2$. For each $(\mathbf{x}, u)$, $\mathbf{f}(\mathbf{x}, u)$ represents a tangent to vector to $\mathbf{R}^2$ at $\mathbf{x}$. Thus the mapping from $\mathbf{R}$ to $T_{\mathbf{x}}\mathbf{R}^2$ defined by $u \mapsto \mathbf{f}(\mathbf{x}, u)$ is a curve in the tangent space to $\mathbf{x}$. Therefore the mapping $V\colon \mathbf{R}^2 \times \mathbf{R} \to T\mathbf{R}^2$ defined by $(\mathbf{x}, u) \mapsto (\mathbf{x}, \mathbf{f}(\mathbf{x}, u))$ gives a submanifold of the tangent bundle whose intersection with each tangent space is a curve parametrized by u.

Let $(\bar{\mathbf{x}}, \bar{u}) = \mathbf{\Phi}(\mathbf{x}, u) = (\boldsymbol{\phi}(\mathbf{x}), \psi(\mathbf{x}, u))$ be a feedback transformation and apply it to the control system (4). Using the chain rule we compute the new control system

$$\bar{\mathbf{f}}(\bar{\mathbf{x}}, \bar{u}) = \frac{d\bar{\mathbf{x}}}{dt} = \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}} \cdot \mathbf{f}(\mathbf{x}, u),$$

and since $\boldsymbol{\phi}(\mathbf{x})$ can be an arbitrary diffeomorphism, we see that the curve $\mathbf{f}(\mathbf{x}, u)$ can be acted on by an arbitrary GL(2,$\mathbf{R}$) matrix. Further, since $\bar{u} = \psi(\mathbf{x}, u)$, we see that feedback allows the curve $\mathbf{f}(\mathbf{x}, u)$ to be reparametrized. Thus the action on the curve is a centro-affine action, and we should expect to see centro-affine invariants in the feedback invariants of the control system (4).

To make effective use of the previous section, we introduce the frame bundle on $\mathbf{R}^2$. Let $\mathcal{F}$ denote the bundle of frames on $\mathbf{R}^2$. A point $e \in \mathcal{F}$ is a linear isomorphism from a tangent space $T_{\mathbf{x}}\mathbf{R}^2$ to $\mathbf{R}^2$. The mapping $e \mapsto \mathbf{x}$ is the projection from $\mathcal{F}$ to $\mathbf{R}^2$. The point e determines a triple $(\mathbf{x}, \mathbf{e}_0, \mathbf{e}_1)$, where $(\mathbf{e}_0, \mathbf{e}_1)$ is the basis of $T_{\mathbf{x}}\mathbf{R}^2$ that e maps to the standard basis of $\mathbf{R}^2$. In other words, $e(\mathbf{e}_0) = {}^t(1,0)$ and $e(\mathbf{e}_1) = {}^t(0,1)$. We may view $\mathbf{x}, \mathbf{e}_0$ and $\mathbf{e}_1$ as functions from $\mathcal{F}$ to $\mathbf{R}^2$. The derivatives of these maps determine the *structure equations* of $\mathcal{F}$,

$$
\begin{aligned}
d\mathbf{x} &= \mathbf{e}_0\sigma^0 + \mathbf{e}_1\sigma^1 \\
d\mathbf{e}_0 &= \mathbf{e}_0\omega^0{}_0 + \mathbf{e}_1\omega^1{}_0 \\
d\mathbf{e}_1 &= \mathbf{e}_0\omega^0{}_1 + \mathbf{e}_1\omega^1{}_1 \\
d\sigma^0 &= -\omega^0{}_0 \wedge \sigma^0 - \omega^0{}_1 \wedge \sigma^1 \\
d\sigma^1 &= -\omega^1{}_0 \wedge \sigma^0 - \omega^1{}_1 \wedge \sigma^1 \\
d\omega^i{}_j &= -\omega^i{}_0 \wedge \omega^0{}_j - \omega^i{}_1 \wedge \omega^1{}_j \quad 0 \le i, j \le 1.
\end{aligned}
\tag{5}
$$

The six 1-forms $\sigma^0, \sigma^1, \omega^0{}_0, \omega^0{}_1, \omega^1{}_0, \omega^1{}_1$ form a basis for the 1-forms on $\mathcal{F}$. Notice that the structure equations for $\mathbf{e}_0$, $\mathbf{e}_1$ and $\omega^0{}_0, \omega^0{}_1, \omega^1{}_0, \omega^1{}_1$ agree with (1) and reflect the GL(2,$\mathbf{R}$), or centro-affine, structure on each tangent space.

Zeroth order frames. The first step for centro-affine curves is to let $\mathbf{e}_0$ be the position vector of the curve. We apply the same technique here and define the zeroth order frame bundle $\mathcal{F}^{(0)}_{\mathbf{f}}$ to be the set of all frames with $\mathbf{e}_0 = \mathbf{f}(\mathbf{x}, u)$. Thus a zeroth order frame can be viewed as a triple $(\mathbf{x}, \mathbf{f}(\mathbf{x}, u), \mathbf{e}_1)$, where $\mathbf{f}$ and $\mathbf{e}_1$ are linearly independent. Notice that the formula for $d\mathbf{x}$ in (5) becomes $d\mathbf{x} = \mathbf{f}\,\sigma^0 + \mathbf{e}_1\sigma^1$, which implies, on using $\langle , \rangle$ to denoting the pairing of vectors with forms, that $\langle \mathbf{f}, \sigma^0 \rangle = 1$ and $\langle \mathbf{f}, \sigma^1 \rangle = 0$.

We wish now to compute the centro-affine invariants of the curves in each tangent space. Since a tangent space is determined by setting $\mathbf{x}$ equal to a constant, all we have to do is compute the derivatives modulo dx^1 and dx^2.

Since $\mathbf{e}_0 = \mathbf{f}$, the equation for $d\mathbf{e}_0$ in (5) shows that $\omega^0{}_0$ and $\omega^1{}_0$ are multiples of du, dx^1 and dx^2. We therefore know that $\omega^0{}_0 \equiv m^0 du$ and $\omega^1{}_0 \equiv m^1 du$ modulo dx^1, dx^2. We would like to choose $\mathbf{e}_1$ so that it points in the same direction as $\partial\mathbf{f}/\partial u$, but since $\mathbf{e}_0$ and $\mathbf{e}_1$ must be linearly independent, we can only do that if $\mathbf{e}_0 = \mathbf{f}$ and $\partial\mathbf{f}/\partial u$ are linearly independent. Being independent is equivalent to $\omega^1{}_0$ being nonzero modulo dx^1, dx^2, which is the same as the function m^1 being nonzero. If m^1 is identically zero, then $\mathbf{f}(\mathbf{x}, u)$ lies on a ray emanating from the origin of $T_{\mathbf{x}}\mathbf{R}^2$, for each $\mathbf{x} \in \mathbf{R}^2$. Thus there is a feedback transformation taking $\mathbf{f}(\mathbf{x}, u)$ to a control linear normal form $\mathbf{g}(\mathbf{y})\,v$, where $(\mathbf{y}, v) = \boldsymbol{\Phi}(\mathbf{x}, u)$.

First order frames. We will assume that m^1 is always nonzero. Then we may define the first order frames $\mathcal{F}^{(1)}_{\mathbf{f}}$ by requiring $\mathbf{e}_1$ to point in the same direction as

$\partial\mathbf{f}/\partial u$, which implies that $\omega^0{}_0 \equiv 0 \pmod{dx^1, dx^2}$. As in the curve case, this condition implies that $\omega^0{}_1 \equiv a\,\omega^1{}_0 \pmod{dx^1, dx^2}$. Differentiation shows that a is acted on by a square, and we have three cases. If a is identically zero, then $\mathbf{f}(\mathbf{x}, u)$ lies on an affine line in each tangent space and thus there is a feedback transformation taking $\mathbf{f}(\mathbf{x}, u)$ to a control linear normal form $\mathbf{g}_0(\mathbf{y}) + \mathbf{g}_1(\mathbf{y})\,v$. We will now consider the final case.

Second order frames. If a is never zero, then we may define the second order frames $\mathcal{F}^{(2)}_{\mathbf{f}}$ by choosing $\mathbf{e}_1$ so that $a = -\epsilon$, where $\epsilon = \pm 1$. This determines the centro-affine arc-length in each tangent space. In fact, using the arc-length formula derived in the previous section, we see that

$$\omega^1{}_0 \equiv \sqrt{\left|\frac{\det(\frac{\partial^2\mathbf{f}}{\partial u^2}, \frac{\partial\mathbf{f}}{\partial u})}{\det(\mathbf{f}, \frac{\partial\mathbf{f}}{\partial u})}\right|}\, du \qquad \pmod{dx^1, dx^2}.$$

Comparing again with centro-affine curves, we see that $\omega^1{}_1$ must be congruent to a multiple of $\omega^1{}_0$, say $\omega^1{}_1 \equiv \kappa(\mathbf{x}, u)\,\omega^1{}_0 \pmod{dx^1, dx^2}$, and for each tangent space $T_{\mathbf{x}}\mathbf{R}^2$, $\kappa(\mathbf{x}, u)$ is the centro-affine curvature of $\mathbf{f}(\mathbf{x}, u)$.

Taken together, we see that we may choose frames adapted to the control system $\mathbf{f}$ so that in each tangent space, the frames satisfy the structure equations for centro-affine plane curves. We can summarize the above congruences with the matrix equation

$$\begin{pmatrix} \omega^0{}_0 & \omega^0{}_1 \\ \omega^1{}_0 & \omega^1{}_1 \end{pmatrix} \equiv \begin{pmatrix} 0 & -\epsilon \\ 1 & \kappa(\mathbf{x}, u) \end{pmatrix} \omega^1{}_0 \qquad \pmod{dx^1, dx^2}.$$

Comparing the expression for $d\mathbf{x}$ in equations (5) with the expression

$$d\mathbf{x} = \frac{\partial}{\partial x^1}\, dx^1 + \frac{\partial}{\partial x^2}\, dx^2$$

we see that σ^0, σ^1 must be a nonsingular linear combination of dx^1, dx^2. Thus any congruence modulo dx^1, dx^2 is equivalent to a congruence modulo σ^0, σ^1. We can therefore rewrite the last equation as

$$\begin{pmatrix} \omega^0{}_0 & \omega^0{}_1 \\ \omega^1{}_0 & \omega^1{}_1 \end{pmatrix} \equiv \begin{pmatrix} 0 & -\epsilon \\ 1 & \kappa(\mathbf{x}, u) \end{pmatrix} \omega^1{}_0 \qquad \pmod{\sigma^0, \sigma^1}. \tag{6}$$

Equation (6) shows how $\mathbf{f}$ varies in a given tangent space, but we also need to know how $\mathbf{f}$ varies as we move from tangent space to tangent space, that is as we vary the state variables $\mathbf{x}$. We do so by putting equation (6) back into the derivative equations for $d\sigma^0, d\sigma^1$ in (5). Writing these equations in matrix form, we have

$$\begin{pmatrix} d\sigma^0 \\ d\sigma^1 \end{pmatrix} = -\begin{pmatrix} 0 & -\epsilon \\ 1 & \kappa(\mathbf{x}, u) \end{pmatrix} \omega^1{}_0 \wedge \begin{pmatrix} \sigma^0 \\ \sigma^1 \end{pmatrix} + \begin{pmatrix} A\,\sigma^0 \wedge \sigma^1 \\ B\,\sigma^0 \wedge \sigma^1 \end{pmatrix}.$$

A short calculation shows that by adding terms to $\omega^1{}_0$ we can rewrite this as

$$\begin{pmatrix} d\sigma^0 \\ d\sigma^1 \end{pmatrix} = -\begin{pmatrix} 0 & -\epsilon \\ 1 & \kappa(\mathbf{x}, u) \end{pmatrix} [\omega^1{}_0 + \epsilon\, A\, \sigma^0 + (B + \epsilon\, \kappa\, A)\, \sigma^1] \wedge \begin{pmatrix} \sigma^0 \\ \sigma^1 \end{pmatrix}. \tag{7}$$

Notice that equation (6), which was arrived at by considering the centro-affine structure of $\mathbf{f}$ on each tangent space, only defines $\omega^1{}_0$ up to linear combinations of

σ^0 and σ^1. Thus we may add such linear terms to ${\omega^1}_0$ and preserve equation (6), equivalently we preserve the centro-affine structure equations for $\mathbf{f}$. We therefore define a new 1-form, $\mu = {\omega^1}_0 + \epsilon\, A\, \sigma^0 + (B + \epsilon\, \kappa\, A)\, \sigma^1$. Substituting into (7) gives

$$\begin{pmatrix} d\sigma^0 \\ d\sigma^1 \end{pmatrix} = - \begin{pmatrix} 0 & -\epsilon \\ 1 & \kappa(\mathbf{x}, u) \end{pmatrix} \mu \wedge \begin{pmatrix} \sigma^0 \\ \sigma^1 \end{pmatrix}. \tag{8}$$

It is easy to see that equation (8) determines μ uniquely. Suppose $\bar{\mu}$ also satisfied equation (8). Then setting the two expressions equal and subtracting gives two equations, $\epsilon(\bar{\mu} - \mu) \wedge \sigma^1 = 0$ and $(\bar{\mu} - \mu) \wedge \sigma^0 + \kappa(\bar{\mu} - \mu) \wedge \sigma^1 = 0$. The first equations implies that there is a function p such that $\bar{\mu} - \mu = p\, \sigma^1$. Substituting into the second equation gives $p\, \sigma^1 \wedge \sigma^0 = 0$, thus $p = 0$ and $\bar{\mu} = \mu$.

We are now at an important point. The three independent 1-forms, σ^0, σ^1 and μ, form a basis for the 1-forms on state-control space, $\mathbf{R}^2 \times \mathbf{R}$. Moreover, this basis remains invariant under the action of feedback transformations. We see this in the following way. To each control system, we assign the centro-affine "Frenet frame" $(\mathbf{e}_0, \mathbf{e}_1)$ in each tangent space. Since a feedback transformation induces a centro-affine action in each tangent space, this frame is feedback invariant. The 1-forms σ^0, σ^1 are dual to the frame $(\mathbf{e}_0, \mathbf{e}_1)$, and are defined by the equation $d\mathbf{x} = \mathbf{e}_0\, \sigma^0 + \mathbf{e}_1\, \sigma^1$. Since the frame is feedback invariant, so are the 1-forms σ^0 and σ^1. Equation (8) determines the 1-form μ. Since feedback preserves σ^0 and σ^1, feedback must also preserve equation (8). Therefore, since equation (8) determines μ uniquely, feedback must also preserve μ.

Using (8) we compute that $0 = d^2\sigma^0 = \epsilon\, d\mu \wedge \sigma^1$, thus $d\mu$ has the form

$$d\mu = -K\, \sigma^0 \wedge \sigma^1 - J\, \mu \wedge \sigma^1.$$

For any function $h(x^1, x^2, u)$, the functions h_{σ^0}, h_{σ^1} and h_μ are uniquely defined by the equation $dh = h_{\sigma^0}\, \sigma^0 + h_{\sigma^1}\, \sigma^1 + h_\mu\, \mu$. With this notation and with equation (8) we compute

$$0 = d^2\sigma^1 = J\, \mu \wedge \sigma^1 \wedge \sigma^0 - d\kappa \wedge \mu \wedge \sigma^1 = (J - \kappa_{\sigma^0})\mu \wedge \sigma^1 \wedge \sigma^0.$$

From this we see that J is a derivative of κ. Finally, computing $d^2\mu = 0$ gives the relation

$$J_{\sigma^0} + \kappa\, K - K_\mu = 0.$$

The above discussion leads to the following structure theorem for two state, single input systems.

Theorem. *There are three regular classes for the two state, single input control system (4).*

- *If $\mathbf{f}(\mathbf{x}, u)$ and $\frac{\partial \mathbf{f}}{\partial u}(\mathbf{x}, u)$ are linearly dependent for all $(\mathbf{x}, u)$, then in each tangent space $\mathbf{f}(\mathbf{x}, u)$ lies on a ray emanating from the origin and is therefore feedback equivalent to the form $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})\, v$.*
- *If $\mathbf{f}(\mathbf{x}, u)$ and $\frac{\partial \mathbf{f}}{\partial u}(\mathbf{x}, u)$ are linearly independent and $\frac{\partial \mathbf{f}}{\partial u}(\mathbf{x}, u)$ and $\frac{\partial^2 \mathbf{f}}{\partial u^2}(\mathbf{x}, u)$ are linearly dependent for all $(\mathbf{x}, u)$, then in each tangent space $\mathbf{f}(\mathbf{x}, u)$ lies on an affine line and is therefore feedback equivalent to the form $\dot{\mathbf{y}} = \mathbf{g}_0(\mathbf{y}) + \mathbf{g}_1(\mathbf{y})\, v$.*
- *If $\mathbf{f}(\mathbf{x}, u)$ and $\frac{\partial \mathbf{f}}{\partial u}(\mathbf{x}, u)$ are linearly independent and $\frac{\partial \mathbf{f}}{\partial u}(\mathbf{x}, u)$ and $\frac{\partial^2 \mathbf{f}}{\partial u^2}(\mathbf{x}, u)$ are linearly independent for all $(\mathbf{x}, u)$, then in each tangent space we may assign a centro-affine frame $(\mathbf{e}_0, \mathbf{e}_1)$ to $\mathbf{f}$. This frame uniquely determines dual 1-forms σ^0 and σ^1 satisfying the equation $d\mathbf{x} = \mathbf{e}_0\, \sigma^0 + \mathbf{e}_1\, \sigma^1$. The*

1-form μ together with $\epsilon = \pm 1$ and the function κ are uniquely determined by equation (8). They additionally satisfy the equations

$$\left.\begin{aligned} d\mathbf{e}_0 &\equiv \qquad\qquad \mathbf{e}_1\,\mu \\ d\mathbf{e}_1 &\equiv -\mathbf{e}_0\,\epsilon\,\mu + \mathbf{e}_1\,\kappa\,\mu \end{aligned}\right\} \qquad (\mathrm{mod}\ \sigma^0, \sigma^1), \tag{9}$$

which imply, when we restrict to a fixed tangent space on the states, that μ is the centro-affine arc-length and κ is the centro-affine curvature of $\mathbf{f}$. Finally, the derivative of μ has the form

$$d\mu = -K\,\sigma^0 \wedge \sigma^1 - J\,\mu \wedge \sigma^1, \tag{10}$$

and the functions κ, K and J are a fundamental set of feedback invariant functions on state-control space. These functions must further satisfy the integrability relations

$$\begin{aligned} 0 &= J - \kappa_{\sigma^0} \\ 0 &= J_{\sigma^0} + \kappa\,K - K_\mu. \end{aligned} \tag{11}$$

Writing equations (8) and (10) as

$$\begin{aligned} d\sigma^0 &= \epsilon\,\mu \wedge \sigma^1 \\ d\sigma^1 &= -\mu \wedge \sigma^0 - \kappa\,\mu \wedge \sigma^1 \\ d\mu &= -K\,\sigma^0 \wedge \sigma^1 - J\,\mu \wedge \sigma^1, \end{aligned}$$

we see that they agree (up to sign) with the structure equations that R. Gardner derives using Cartan's equivalence method [**Ga89**]. (Comparing with Gardner's equation, $\eta^1 = \sigma^0$, $\eta^2 = \sigma^1$, and $I = -\kappa$.) We arrived at the *same* set of structure equations, and *entirely bypassed* the equivalence method. Moreover, we have a *geometric interpretation* of the quantities. The 1-form μ represents the centro-affine arc-length of $\mathbf{f}$ in each tangent space, and $\kappa(\mathbf{x}, u)$ represents the centro-affine curvature in each tangent space. The sign of ϵ determines the convexity of $\mathbf{f}$. The functions $K(\mathbf{x}, u)$ and $J(\mathbf{x}, u)$ describe the variation of $\mathbf{f}$ from one tangent space to another.

3. Examples

The fundamental invariants are constant. Suppose now that κ, K and J are constant. We see from (11) that J must be 0 and $\kappa\,K = 0$. We will consider the case $\kappa \neq 0$ first. Thus $K = 0$ and $\epsilon = \pm 1$. We will examine the case $\epsilon = -1$, the other case being similar. We begin by recalling that a curve in the centro-affine plane with structure equations (3), $\epsilon = -1$, and κ equal to a constant is equivalent to the standard form $\mathbf{x}(s) = {}^t(1, 0)\,e^{\lambda_1 s} + {}^t(0, 1)\,e^{\lambda_2 s}$, where λ_1 and λ_2 are the roots of $r^2 - \kappa\,r - 1 = 0$. We arrived at this form by using the fact that equations (3) imply that $\mathbf{x}(s)$ satisfies the differential equation $\mathbf{x}''(s) - \kappa\,\mathbf{x}'(s) - \mathbf{x}(s) = 0$. In this form we see that $\kappa = \lambda_1 + \lambda_2$. The equivalent normal form

$$\mathbf{x}(s) = \left[\begin{pmatrix} \cosh(\sqrt{\kappa^2 + 4}\,s/2) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sinh(\sqrt{\kappa^2 + 4}\,s/2) \end{pmatrix}\right] e^{\kappa\,s/2}$$

has the advantage that the constant κ is easier to read off. What we shall show is that, under the current assumptions, the control system $\mathbf{f}(\mathbf{x}, u)$ is feedback equivalent to the nonlinear system

$$\begin{aligned} \mathbf{g}(\mathbf{y}, v) = \mathbf{g}(v) &= \left[\begin{pmatrix} \cosh(\sqrt{\kappa^2+4}\, v/2) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sinh(\sqrt{\kappa^2+4}\, v/2) \end{pmatrix} \right] e^{\kappa\, v/2} \\ &= \begin{pmatrix} \cosh(\sqrt{\kappa^2+4}\, v/2) \\ \sinh(\sqrt{\kappa^2+4}\, v/2) \end{pmatrix} e^{\kappa\, v/2}. \end{aligned} \tag{12}$$

Since J and K vanish, the structure equations (8) and (10) are

$$\begin{aligned} d\sigma^0 &= -\mu \wedge \sigma^1 \\ d\sigma^1 &= -\mu \wedge \sigma^0 - \kappa\, \mu \wedge \sigma^1 \\ d\mu &= 0. \end{aligned} \tag{13}$$

We see that there is a function $v = v(\mathbf{x}, u)$, defined up to a constant, such that $\mu = dv$. Choose $v(\mathbf{x}, u)$ so that 0 is in its range. Since σ^0, σ^1 and μ are independent, and since σ^0 and σ^1 are linear combinations of dx^1 and dx^2 only, we see that $\partial v/\partial u \neq 0$. Recalling that $\mathbf{e}_0 = \mathbf{f}(\mathbf{x}, u)$, equations (9) imply that $\mathbf{e}_1 = \partial \mathbf{f}/\partial v$ and $\partial^2 \mathbf{f}/\partial v^2 - \kappa\, \partial \mathbf{f}/\partial v - \mathbf{f} = 0$. We will now restrict to the two dimensional submanifold defined by $v = 0$. Equations (13) show that, along $v = 0$, the Lie bracket $[\mathbf{e}_0, \mathbf{e}_1] = 0$. Thus, we may apply a change of state variable, $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$ such that

$$\begin{aligned} \mathbf{e}_0|_{v=0} &= \frac{\partial}{\partial y^1} \\ \mathbf{e}_1|_{v=0} &= \frac{\kappa}{2} \frac{\partial}{\partial y^1} + \frac{\sqrt{\kappa^2+4}}{2} \frac{\partial}{\partial y^2}. \end{aligned} \tag{14}$$

Together, $\mathbf{y}$ and v determine a feedback transformation that transforms $\mathbf{f}(\mathbf{x}, u)$ to $\mathbf{g}(\mathbf{y}, v)$, where $\mathbf{g}(\mathbf{y}, 0) = \partial/\partial y^1$ and $\partial \mathbf{g}/\partial v(\mathbf{y}, 0) = \kappa/2\, \partial/\partial y^1 + \sqrt{\kappa^2+4}/2\, \partial/\partial y^2$. Moreover, since all of our centro-affine structure is preserved under feedback, we must also have that $\mathbf{g}$ satisfies the same differential equation as $\mathbf{f}$, namely for all $\mathbf{y}$, $\partial^2 \mathbf{g}/\partial v^2 - \kappa\, \partial \mathbf{g}/\partial v - \mathbf{g} = 0$, with the initial conditions at $v = 0$ given above. The function given in (12) satisfies this differential equation and these initial conditions, so $\mathbf{g}(\mathbf{y}, v)$ must be equal to (12). Thus we have a normal form for control systems of this type.

If $\epsilon = +1$ and $\kappa \neq 0$, a similar argument gives one of three normal forms. The particular form depends on whether $\kappa^2 - 4$ is positive, zero or negative. We find that $\mathbf{f}(\mathbf{x}, u)$ is respectively equivalent to either

- $\mathbf{g}(\mathbf{y}, v) = {}^t(\cosh[\sqrt{\kappa^2-4}\, v/2], \sinh[\sqrt{\kappa^2-4}\, v/2]) e^{\kappa\, v/2}$,
- $\mathbf{g}(\mathbf{y}, v) = {}^t(1, v) e^{\pm v}$, or
- $\mathbf{g}(\mathbf{y}, v) = {}^t(\cos[\sqrt{4-\kappa^2}\, v/2], \sin[\sqrt{4-\kappa^2}\, v/2]) e^{\kappa\, v/2}$.

The case $\boldsymbol{\kappa} = 0$. With $\kappa = 0$, we still must have that $J = 0$, but K can be nonzero. Equations (8) and (10) become

$$\begin{aligned} d\sigma^0 &= \epsilon\, \mu \wedge \sigma^1 \\ d\sigma^1 &= -\mu \wedge \sigma^0 \\ d\mu &= -K\, \sigma^0 \wedge \sigma^1. \end{aligned} \tag{15}$$

We recall that the centro-affine curvature of a plane curve vanishes if and only if the curve lies on a quadric (convex when $\epsilon = 1$, concave when $\epsilon = -1$). Since we are assuming that $\kappa = 0$, we see that the control system $\mathbf{f}(\mathbf{x}, u)$ determines a quadric curve in each tangent space. We can use this curve to define the length of a tangent vector. We do this by declaring $\|\mathbf{f}(\mathbf{x}, u)\|$ to be 1 for all $(\mathbf{x}, u)$. This suggests that we have either a Riemannian or a pseudo-Riemannian metric on the state space. In fact, we can show that this metric is given by $\|\mathbf{w}\|^2 = \sigma^0(\mathbf{w})^2 + \epsilon\,\sigma^1(\mathbf{w})^2$, for any $\mathbf{w} \in T_{\mathbf{x}}\mathbf{R}^2$. We have to be a bit careful here, because the 1-forms σ^0 and σ^1 are defined on $(\mathbf{x}, u)$-space, and we are claiming that the quadratic combination $(\sigma^0)^2 + \epsilon\,(\sigma^1)^2$ is well defined on $\mathbf{x}$-space. To show this, let Z be any vector field tangent to the fiber of $(\mathbf{x}, u)$-space over $\mathbf{x}$ space, that is, tangent to the fiber of state-control space over state space. Since σ^0, σ^1 are nonsingular linear combinations of dx^1, dx^2, we see that every such vector Z is defined by the equations $\langle Z, \sigma^0\rangle = \langle Z, \sigma^1\rangle = 0$. Using Cartan's formula for Lie derivative, $\mathcal{L}_Z\,\theta = Z \lrcorner\, d\theta + d(Z \lrcorner\, \theta)$, and using equations (15) we can compute the Lie derivative $\mathcal{L}_Z\,(\sigma^0)^2 + \epsilon\,(\sigma^1)^2 = 2(\sigma^0\mathcal{L}_Z\,\sigma^0 + \epsilon\,\sigma^1\mathcal{L}_Z\,\sigma^1) = 2\langle Z, \mu\rangle(\epsilon\,\sigma^0\,\sigma^1 - \epsilon\,\sigma^1\,\sigma^0) = 0$. Since Z is an arbitrary vertical vector field, this shows that $(\sigma^0)^2 + \epsilon\,(\sigma^1)^2$ is a well defined metric on state space. Moreover, we can interpret our control system $\mathbf{f}(\mathbf{x}, u)$ as simply saying that we move along the state space at unit speed.

Since the 1-forms σ^0 and σ^1 diagonalize the Riemannian metric, we recognize that equations (15) are simply the structure equations of a surface, with μ representing the Levi-Civita connection 1-form and K representing the Gaussian curvature of the surface. We see instantly that if K is constant, then our control system must be equivalent to a control system that parametrizes unit speed curves on constant curvature surfaces. For example, in the $\epsilon = 1$ case, if $(\sigma^0)^2 + (\sigma^1)^2$ is isometricly equivalent to the constant curvature metric

$$\frac{(dy^1)^2 + (dy^2)^2}{\left(1 + \frac{K}{4}\left[(y^1)^2 + (y^2)^2\right]\right)^2}$$

then the control system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$ must be feedback equivalent to the control system

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, v) = \begin{pmatrix} \cos v \\ \sin v \end{pmatrix} \left(1 + \frac{K}{4}\left[(y^1)^2 + (y^2)^2\right]\right),$$

since this is equivalent to the metric norm being equal to one, $\|\dot{\mathbf{y}}\|^2 = 1$.

Constant κ in each fiber. Suppose now that in each fiber, $\mathbf{f}(\mathbf{x}, u)$ defines a constant curvature centro-affine curve, but that the constant can vary as we move to different fibers. In other words, assume that κ is a nonconstant function depending only on $\mathbf{x}$. We can construct examples of this kind of system by taking a normal form such as (12) and allow κ to be a function of $\mathbf{x}$. For this kind of system, we can see that J is arrived at in a very natural way. Since κ only depends on $\mathbf{x}$, $\kappa_\mu = 0$ and therefore $d\kappa = \kappa_{\sigma^0}\,\sigma^0 + \kappa_{\sigma^1}\,\sigma^1$. Since $\langle \mathbf{f}, \sigma^0\rangle = 1$ and $\langle \mathbf{f}, \sigma^1\rangle = 0$, we see that $\langle \mathbf{f}, d\kappa\rangle = \kappa_{\sigma^0}$, but $\kappa_{\sigma^0} = J$ by (11), so we see that J is simply the derivative of κ by the control vector field $\mathbf{f}$. So, in this case at least, we have a nice geometric interpretation of J. It is giving us the rate at which the centro-affine curvature changes as we move from fiber to fiber along the state space in the direction of our control.

A Gauss-Bonnet theorem. Suppose that in each fiber $\mathbf{f}(\mathbf{x}, u)$ defines a strictly convex simple closed curve surrounding the origin. Then we may view these curves as the unit vectors, or indicatrix, for a Finsler metric on state space. Equations (8) and (10) are the structure equations for a Finsler surface. We have already seen that the vanishing of κ means that the Finsler metric is in fact a Riemannian metric, K is the Gaussian curvature and μ is the connection 1-form.

Let $S_{\mathbf{x}} \subset T_{\mathbf{x}}\mathbf{R}^2$ denote the indicatrix at $\mathbf{x}$. The centro-affine perimeter of $S_{\mathbf{x}}$ is given by

$$P_{\mathbf{x}} = \int_{S_{\mathbf{x}}} \mu.$$

Computing the variational derivative $\delta \int \mu = 0$ shows that $P_{\mathbf{x}}$ is constant exactly when the invariant J vanishes. These are the Landsberg surfaces. Assuming that J vanishes, equation (10) gives

$$d\mu = -K\, \sigma^0 \wedge \sigma^1.$$

From this equation we see that the 2-form $K\, \sigma^0 \wedge \sigma^1$ is well defined on state space. We also see that μ is a transgression for $K\, \sigma^0 \wedge \sigma^1$ on state-control space. These observations show how we may directly translate the proofs of the standard Gauss-Bonnet theorems into this setting, with the constant perimeter $P_{\mathbf{x}}$ replacing 2π.

4. Closing

When we view a control system as a family of submanifolds of the linear tangent spaces to the state space, we see that the action of the feedback group on each unparametrized submanifold is exactly a centro-affine action. This observation suggests that we take a look at centro-affine geometry, that is, the geometry of submanifolds of $\mathbf{R}^n$ under the action of GL(n,$\mathbf{R}$), in the hope that we will gain geometric insight into the overall geometry of the control system.

We studied the very simplest case in this paper, that is the case of curves in the plane. These dimensions correspond exactly to the case of a single input control system on the plane. More generally, centro-affine curves in $\mathbf{R}^n$ correspond to single input control systems on $\mathbf{R}^n$. We gave a complete description of the invariants of centro-affine plane curves, along with a collection of normal forms. We carried this understanding of centro-affine curves over to the two state, single input control problem. The strength of our approach is that it allows us to analyze very nonlinear systems on the plane. In fact, the system must be so nonlinear that is cannot even lie on an affine line in each tangent space. Another advantage to this approach is that we find the same fundamental invariants and structure equations we are led to using Cartan's method of equivalence. Moreover, we gain a geometric interpretation of the invariants and of the structure equations that is difficult to discern from the equivalence method.

We found normal forms for several nonlinear systems, as well as a class of systems possessing a Riemannian metric. In fact, the presence of this metric, which corresponds to the vanishing of κ, is easily predicted from centro-affine considerations. We know that for centro-affine curves, the vanishing of κ is equivalent to having a quadric curve centered at the origin. Knowing this, the vanishing of the feedback invariant κ gives a quadric curve in each tangent space. We may use these curves as a set of unit vectors in each tangent space, which defines the metric on state space. The control system itself may then be interpreted as requiring that a particle move through state space at unit speed.

To extend this study to higher dimensions, we need to study higher dimensional centro-affine geometries. This has been done in part. We have computed the structure of centro-affine hypersurfaces in $\mathbf{R}^n$ and applied these results to feedback control [**GW96**], [**Wi96**]. We have also computed the structure of centro-affine curves in $\mathbf{R}^n$ [**GW97**] and applications to feedback control are under way.

References

Bl23. W. Blaschke, *Vorlesungen über Differentialgeometrie. Affine Differentialgeometrie*, vol. 2, Springer-Verlag, Berlin, 1923.

Ca51. É. Cartan, *La théorie des groupes finis et continus et la géométrie différentielle*, Gauthier-Villars, Paris, 1951.

Ga89. R.B. Gardner, *The Method of Equivalence and Its Applications*, vol. 58, SIAM, Philadelphia, PA, 1989.

GW96. R.B. Gardner and G.R. Wilkens, *A pseudo-group isomorphism between control systems and certain generalized Finsler structures*, Finsler Geometry (Seattle, WA, July 1995) (D. Bao, S.S. Chern and Z. Shen, eds.), vol. Contemporary Math. 196, 1996, pp. 231–243.

GW97. R.B. Gardner and G.R. Wilkens, *The fundamental theorems of curves and hypersurfaces in centro-affine geometry*, Bull. Belgian Math. Society **4** (1997), no. 3, 379–401.

Gn78. M.L. Green, *The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces*, Duke Mathematical Journal **45** (1978), 735–779.

Gr74. P.A. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Mathematical Journal **41** (1974), 775–814.

MM33. O. Mayer and A. Myller, *La géométrie centroaffine des courbes planes*, Annales Scíentifíques de l'Université de Jassy **18** (1933), 234–280.

SS62. P. Schirokow and A. Schirokow, *Affine Differentialgeometrie*, B.G. Teubner Verlagsgesellschaft, Leipzig, 1962.

Sh96. R.W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Programme*, vol. 166, Springer-Verlag, 1996.

Wi96. G.R. Wilkens, *Finsler geometry in low dimensional control theory*, Finsler Geometry (Seattle, WA, July 1995) (D. Bao, S.S. Chern and Z. Shen, eds.), vol. Contemporary Math. 196, 1996, pp. 245–257.

Fields Institute for Research in Mathematical Sciences, 185 Columbia Street West, Waterloo, Ontario N2L 5Z5 CANADA

Current address: Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI 96822

E-mail address: `grw@math.hawaii.edu`