A PSEUDO-GROUP ISOMORPHISM BETWEEN CONTROL SYSTEMS AND CERTAIN GENERALIZED FINSLER STRUCTURES

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Abstract

The equivalence problem for control systems under non-linear feedback is recast as a problem involving the determination of the invariants of submanifolds in the tangent bundle of state space under fiber preserving transformations. This leads to a fiber geometry described by the invariants of submanifolds under the general linear group, which is the classical subject of centro-affine geometry.

Similarly the equivalence problem for Finsler structures is shown to lead to a fiber geometry over the base inducing a centro-affine geometry. The appearance of a centro-affine fiber geometry in both control systems and Finsler structures will be explained after establishing a canonical pseudo-group isomorphism between arbitrary control systems and certain generalized Finsler geometries, that is variational problems with non-holonomic constraints. The generalized Finsler structure turns out to be the geometry of the constrained variational problem arising from the variational problem of time optimal control along control trajectories.

Further analysis will show that a classical Finsler structure will correspond to regular control systems with m-states and $(m-1)$-controls. The term regular will be made precise in section 3, but some centro-affine geometry is needed for the definition.

The original solution of the feedback equivalence problem for the regular system in m-states and $(m-1)$-controls, due to Robert Bryant and the first author, was sufficiently complicated that a complete proof was never published, although an outline exists in [Ga89]. This approach had the disadvantage that the meanings of even the simplest invariants were not visible.

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Adapting moving frames to both the centro-affine fiber geometry and the geometry induced by a regular system in m-states and $(m-1)$-controls or a corresponding Finsler structure leads to a unique extension of the fiber structure equations across the base space and produces a surprisingly simple solution of both equivalence problems. An outline of the strategy in the control setting appeared in [GaWi93].

In the above setting the fiber geometries are that of centro-affine hypersurfaces, and since there is currently no description of these geometries using forms and moving frames, we include a section outlining those results.

1. Introduction

A *control system* is an underdetermined system of ordinary differential equations

$$\frac{dx}{dt} = f(x,u) \qquad \text{with} \quad x \in \mathbf{R}^m \quad \text{and} \quad u \in \mathbf{R}^n$$

where the x variables are called *states* and the u variables are called *controls*.[1] Such a system is equivalent to its associated Pfaffian system

$$K = \{dx - f dt\}.$$

We will study the problem of the determination of a complete set of invariants of such a system under feedback equivalences, which are the diffeomorphisms of the form

$$\Phi(t,x,u) = (t, \varphi(x), \psi(x,u)),$$

preserving integral curves of the associated Pfaffian system.

A Finsler structure on an m-manifold M is a map

$$F : T(M) \longrightarrow [0,\infty)$$

from the tangent bundle $T(M)$ to the closed half line, differentiable off the zero section and satisfying

$$F(tv) = |t| F(v) \quad \text{for} \;\; t \in \mathbf{R},\; v \in T(M).$$

That is F is homogeneous of degree one in the fibers. In addition, the matrix of second derivatives

$$\left(\frac{\partial^2}{\partial u^i \partial u^j} [F^2(x,u)] \right)$$

[1] We represent x, u, and $f(x,u)$ as row vectors, so matrices and column vectors will multiply to the right.

is positive definite, where the pair (x, u) represents the tangent vector $v_x = \sum u^i \partial/\partial x^i$.

Two Finsler structures F on a manifold M and $\widetilde{F}$ on a manifold $\widetilde{M}$ are *simply equivalent* if there exists a bundle diffeomorphism Φ between their tangent bundles such that $\widetilde{F} \circ \Phi = F$. In other words if the diagram

$$\begin{array}{ccc} T(M) & \xrightarrow{\Phi} & T(\widetilde{M}) \\ {\scriptstyle Id}\downarrow & & {\scriptstyle \tilde{F}}\downarrow \\ T(M) & \xrightarrow{F} & [0,\infty) \end{array}$$

commutes and Φ covers a diffeomorphism $\varphi : M \to \widetilde{M}$.

We will also study the problem of the determination of a complete set of invariants of Finsler structures under simple equivalence.

We will see that both problems have a natural centro-affine geometry in the fibers, a property that will be heavily utilized in our analysis.

2. The Centro-affine Geometry of the Fibers

A study via forms and moving frames of centro-affine hypersurface theory which is essentially needed in the rest of the paper has not yet appeared, hence we include an outline for the convenience of the reader, see [GaWi] for full details and references to other approaches.

Thus we consider the centro-affine geometry of a hypersurface

$$Y : M_{m-1} \longrightarrow \mathbf{R}^m.$$

The hypersurface is called *generic* if it does not pass through the origin and Y is normal at every point.

Now given an affine frame $(e_0, \ldots, e_{m-1})$ at the origin, we have the structure equations

$$de_i = \sum_{j=0}^{m-1} \omega_i^j e_j \quad \text{and} \quad d\omega_i^j = \sum_{k=0}^{m-1} \omega_i^k \wedge \omega_k^j \qquad (0 \le i, j \le m-1).$$

If we now adapt the family of frames so that $e_0 = Y$ and $(e_1, \ldots, e_{m-1})$ are tangent, then it follows immediately that $\omega_0^0 = 0$ and differentiation of this normalization and Cartan's lemma guarantee the existence of a symmetric matrix of functions $(h_{\alpha\beta})$ satisfying

$$\omega_\alpha^0 = \sum_{\beta=1}^{m-1} h_{\alpha\beta}\omega_0^\beta, \qquad (1 \le \alpha \le m-1).$$

If we package this information in the symmetric quadratic differential form

$$II_{CA} = \sum_{\alpha,\beta=1}^{m-1} h_{\alpha\beta}\omega_0^\alpha \odot \omega_0^\beta,$$

then we have the analog of the Blaschke metric in affine geometry.

The hypersurface is *non-degenerate* if $(h_{\alpha\beta})$ is non-degenerate, and *strongly convex* relative to the origin if and only if $(h_{\alpha\beta})$ is negative definite. A hypersurface is *regular* if it is strongly convex and generic.

We assume from now on that the hypersurface is regular. The admissible action on this symmetric tensor $(h_{\alpha\beta})$ includes conjugation and hence may be normalized in the usual ways. Let us restrict to the negative definite case, since this involves the simplest notation. Under this hypothesis we may normalize

$$h_{\alpha\beta} = -\delta_{\alpha\beta}.$$

Integrability conditions then imply that

$$\begin{aligned}\Delta_\alpha^\beta &= \frac{1}{2}(\omega_\beta^\alpha + \omega_\alpha^\beta)\\ &= \sum_{\gamma=1}^{m-1} C_{\alpha\gamma}^\beta \omega_0^\gamma, \quad (1 \le \alpha, \beta \le m-1)\end{aligned}$$

where the symbol $(C_{\alpha\gamma}^\beta)$ is symmetric in all three indices. The resulting cubic form

$$P_{CA} = \sum_{\alpha,\beta,\gamma=1}^{m-1} C_{\beta\gamma}^\alpha \omega_0^\beta \odot \omega_0^\alpha \odot \omega_0^\gamma$$

is the analog of the Pick cubic form in affine geometry.

Now let us introduce a matrix notation to compactify the information in the normalized structure equations. Thus we define

$$\begin{aligned}&\Omega = (\omega_\beta^\alpha), \quad \Phi = \frac{1}{2}(\Omega - {}^t\Omega), \quad \Delta = \frac{1}{2}(\Omega + {}^t\Omega),\\ &\omega = (\omega_0^\alpha), \quad e = {}^t(e_1, \ldots, e_{m-1}),\end{aligned}$$

so that

$$d\begin{pmatrix} e_0 \\ e \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi + \Delta \end{pmatrix}\begin{pmatrix} e_0 \\ e \end{pmatrix}.$$

Differentiation of this last set of equations gives

$$d\begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi+\Delta \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi+\Delta \end{pmatrix} \wedge \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi+\Delta \end{pmatrix},$$

and hence

$d\omega = \omega\wedge\Phi + \omega\wedge\Delta$ with ${}^t\Phi = -\Phi$ and Δ is totally symmetric as defined above.

Now this last set of equations determines Φ uniquely. This is seen by using an algebraic theorem similar to that used to prove the characterization of the Levi-Civita connection in Riemannian geometry.

3. Differential Forms Characterizing Control Systems and Feedback

We can view a control system as the submanifold of the tangent space of the state space $T(\mathbf{R}^m)$ defined by

$$V : \mathbf{R}^m \times \mathbf{R}^n \longrightarrow T(\mathbf{R}^m),$$

where

$$V(x,u) = \sum_{i=1}^{m} f^i(x,u)\frac{\partial}{\partial x^i}.$$

Now, if we fix a fiber over $x \in \mathbf{R}^m$ and restrict the mapping V to define

$$V_x : \mathbf{R}^n \to T_x(\mathbf{R}^m) \simeq \mathbf{R}^m,$$

where $V_x(u) = f(x,u)$, then a feedback equivalence $\tilde{x} = \varphi(x)$ satisfies

$$\frac{d\tilde{x}}{dt} = \frac{dx}{dt}\frac{\partial\tilde{x}}{\partial x} = f\frac{\partial\tilde{x}}{\partial x}.$$

This means that the action induced on the fiber is $\partial\tilde{x}/\partial x$, which is an arbitrary member of the general linear group since φ was an arbitrary diffeomorphism. Thus the fiber geometry is the study of invariants of submanifolds under the general linear group, that is centro-affine geometry. We assume that the image of V_x omits the exceptional orbit given by the origin. This means $f \neq 0$, and hence there are no rest points. Another general centro-affine property is the condition that the position vector V is normal to the image of V. In the control linear systems theory this is equivalent to non-zero drift.

We now describe 1-forms which determine both the control system and its feedback equivalences. The idea is to focus on the basic properties preserved by feedback, which are time, state space, and integral curves of the control system.

In particular the Pfaffian system I given by the set of 1-forms on $\mathbf{R}^m \times \mathbf{R}^n$ which are linear combinations of the dx and vanish on the integral curves of the control system is one such geometric object. This system has a simple direct description

$$I = \{dx\, G \mid G : \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R}^m \text{ and } f\, G = 0\}.$$

This is visible since the condition $f\, G = 0$ is precisely the condition needed in order that $dx\, G = (dx - f dt)\, G$. Since we assume there are no rest points, that is $f \neq 0$, this description also makes it clear that the system I has rank $(m-1)$.

A second such geometric object is the affine Pfaffian system of 1-forms which are linear combinations of the dx having the property that they restrict to dt along any integral curve of the control system. Since the difference of two elements of this affine system vanishes along any integral curve of the control system, that difference lies in I and we see that the affine system is modeled on I and hence can be written in the form $\phi + I$.

An explicit representative for ϕ is given by

$$\phi = \frac{f \cdot dx}{f \cdot f}$$

where $\cdot$ is the usual Euclidean dot product. This structure can also be thought of as a non-zero section of the quotient bundle $T^*(\mathbf{R}^m)/I$.

These two structures are enough to characterize when a diffeomorphism is a feedback equivalence as is made precise in the following theorem contained in the work of Gardner and Shadwick [3]. The theorem was initially proved by utilizing the method of equivalence as described in [1], and this was lengthy. Utilizing the above constructions the proof is now very short.

Theorem 1. *A diffeomorphism $\Phi : \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^n$ leaving t invariant is a feedback equivalence of the control system I if and only if*

$$(1) \quad \Phi^* I = I \quad \textit{and} \quad (2) \quad \Phi^*\phi \in \phi + I.$$

Proof. In particular the Pfaffian system J defined by augmenting I by ϕ is an m-dimensional system made up of linear combinations of dx and hence equals the space of state variables. We note since $J = \{dx\}$, it is completely integrable.

Similarly the Pfaffian system $K = \{dx - f dt\}$, satisfies

$$K = \{\phi - dt, I\},$$

since both systems involve only the differentials in time and state space variables and have the same integral curves, and hence the same annihilators.

The two conditions in the theorem along with the characterization of $\{dx\}$ and $\{dx - f dt\}$ just given, are equivalent to Φ preserving states and integral curves, and this is equivalent to Φ being a feedback equivalence. □

4. Differential Forms Characterizing Finsler Structures and Simple Equivalence

The *indicatrix* at $x \in M$ of a Finsler structure F on M is the locus

$$S_x = \{v_x \in T_x(M) | F(v_x) = 1\}.$$

Note that the indicatrix is strongly convex and centrally symmetric. The unit sphere bundle, $S(M)$, is the union over $x \in M$ of S_x. If Φ is a simple equivalence then

$$\Phi : (S_x) \longrightarrow S_{\varphi(x)}$$

is a diffeomorphism.

For each $v_x \in S(M)$, the natural projection from $S(M)$ to M allows us to view $T_x^*(M)$ as a natural m-dimensional subspace of $T_{v_x}^*(S(M))$. Given a Finsler structure F we define an affine Pfaffian system $\phi + I$ on $S(M)$ by the properties that at each $v_x \in S_x \subset T_x(M)$,

$$\phi + I = \{\zeta \in T_x^*(M) \, |< v_x, \zeta >= 1\},$$

and thus

$$I = \{\omega \in T_x^*(M) \, |< v_x, \omega >= 0\}.$$

The key property of this affine system is that it characterizes unit speed curves. The proof of this property, which is formally stated in the following proposition, is a straightforward application of the definitions.

Proposition. *A curve $\tilde{\sigma}(t)$ in $S(M)$ projects to a regular curve $\sigma(t)$ in M satisfying $\tilde{\sigma}(t) = \sigma'(t)$ if and only if*

$$\tilde{\sigma}^* I = 0 \text{ and } \tilde{\sigma}^* \phi = dt.$$

The position vector of the indicatrices can be written in the form

$$\sum_{k=1}^{m} \frac{u^i}{F} \partial / \partial x^i,$$

and hence a representative for ϕ is given by $\phi = \sum_{k=1}^{m} F_{u^i} dx^i$, since by Euler's theorem

$$< \sum_{k=1}^{m} \frac{u^i}{F} \partial/\partial x^i, \phi >= \frac{1}{F} \sum_{k=1}^{m} u^i F_{u^i} = 1.$$

Note that the indicatrices and $\phi + I$ determine each other and that $\phi + I$ is semi-basic, meaning that only the differentials in the base variables occur in any set of generators. In addition by dimension the Pfaffian system

$$\{I, \phi\} = \{dx\}$$

just as in the control geometry.

Theorem 2. *A simple equivalence of the Finsler structure* $\Phi : T(\mathbf{R}^m) \longrightarrow T(\mathbf{R}^m)$ *restricts to a diffeomorphism* $\Phi : S(\mathbf{R}^m) \longrightarrow S(\mathbf{R}^m)$ *satisfying*

$$(1) \quad \Phi^* I_{\Phi(v)} = I_v \quad \textit{and} \quad (2) \quad \Phi^* \phi_{\Phi(v)} \in \phi_v + I_v \quad \textit{for all} \quad v \in S(M).$$

Moreover, a diffeomorphism $\Phi : S(\mathbf{R}^m) \longrightarrow S(\mathbf{R}^m)$ *satisfying (1) and (2) naturally extends to a simple equivalence of the Finsler structure.*

Proof. Given a simple equivalence Φ, it clearly restricts to a diffeomorphism of corresponding sphere bundles. The above definitions imply

$$\Phi^* I_{\Phi(v)} = I_v \quad \text{and} \quad \Phi^* \phi_{\Phi(v)} \in \phi_v + I_v \quad \text{for all} \quad v \in S(M).$$

Conversely given a diffeomorphism Φ of the sphere bundles satisfying (1) and (2), we have that Φ must preserve the Pfaffian system $\{I, \phi\} = \{dx\}$. Therefore Φ covers a diffeomorphism $\varphi : \mathbf{R}^m \to \mathbf{R}^m$. Using the proposition, a short calculation shows that for every $v_x \in S_x \subset T_x(\mathbf{R}^m)$,

$$\Phi(v_x) = \varphi_*(v_x).$$

Thus Φ is simply the restriction of φ_* to $S(\mathbf{R}^m)$. Since φ_* preserves the unit spheres of the Finsler structure, it is clearly a simple equivalence extending Φ. □

This theorem shows that a simple equivalence must be the derivative of a diffeomorphism on the base, verifying that the action is a centro-affine action. In this case the fiber geometry is the centro-affine geometry of the indicatrix, which is always a strongly convex hypersurface.

5. The Pseudo-group Isomorphism

Theorem 3. *There is a canonical pseudogroup isomorphism between arbitrary control systems and certain generalized Finsler structures which takes feedback equivalences into simple equivalence.*

Proof. By section 3 an arbitrary control system and its feedback equivalences are determined by an affine system of 1-forms $\phi + I$. The mapping

$$\Upsilon : \text{Control Systems} \longrightarrow \text{Generalized Finsler structures}$$

defined by

$$\Upsilon(\phi + I) = \delta \int_I \phi$$

is well defined since the notation means that the variation is over integral curves of I and by Theorem 1 and Theorem 2 carries feedback diffeomorphisms into simple equivalences. □

The natural question of which generalized Finsler structures are in the image has been determined [GaShWi89]. Given

$$\delta \int_I \phi,$$

then the key necessary condition is that the Pfaffian system $J = \{\phi, I\}$ be completely integrable. In order to recover the state space some topological conditions are necessary. The precise statement is as follows.

Theorem 4. *An affine Pfaffian system $\{\phi + I\}$ on a (m+n)- manifold M where the Pfaffian system I is of rank $m - 1$ and ϕ is non-zero such that*

(1) $J = \{\phi, I\}$ is completely integrable,

(2) The leaf space of J on M is a manifold and the natural projection of M on the leaf space $\mathcal{X}$ is a submersion.

Then locally there is a control system with affine system the given $\phi + I$ and with $\mathcal{X}$ the state space.

Proof. See [BrGa]. □

Thus the generalized Finsler structures $\delta \int_I \phi$ which have an associated affine system satisfying this last theorem make precise the meaning of *certain* in Theorem 3.

A natural question is to characterize those control systems which correspond to a Finsler structure. Since the fiber geometries must correspond and the fiber geometry of a Finsler geometry on an m-manifold is the hypersurface geometry of the indicatrix, the fiber geometry of a corresponding control system must also give a hypersurface geometry. This means that such a control system has m-states and $(m-1)$-controls.

We will say that a Finsler structure or a control system with m-states and $(m-1)$-controls is *regular* if the fiber geometry over each point is a regular hypersurface.

We assume from now on that our Finsler structures or control systems with m-states and $(m-1)$-controls are regular.

6. Evolution of the Fiber Geometry across the State Space and the Solution of the Equivalence Problems

We assume that we have either a regular Finsler structure on $\mathbf{R}^m$ or a regular control systems with m-states and $(m-1)$-controls, and begin by choosing a frame $(e_0, e_1, \ldots, e_{m-1})$ on $\mathbf{R}^m$ defined for each point (x, u) in the fiber of $T(R^m)$ over x. This determines a dual coframe $(\omega^0, \omega^1, \ldots, \omega^{m-1})$ on $\mathbf{R}^m$ for each point (x, u) by

$$dx = \omega^0 e_0 + \sum_{\alpha=1}^{m-1} \omega^\alpha e_\alpha .$$

As the reader will see shortly, the first leg will have special meaning, which motivates the curious range of indices.

Now given an integral curve for a regular control system $\gamma : \mathbf{R} \longrightarrow \mathbf{R}^m \times \mathbf{R}^{m-1}$ then

$$f dt = \gamma^* dx = \gamma^* \omega^0 e_0 + \sum_{i=0}^{m-1} \gamma^* \omega^\alpha e_\alpha ,$$

hence choosing $e_0 = f$ results in the conditions

$$\gamma^*\omega^0 = dt, \quad \gamma^*\omega^1 = 0, \ \ldots \ , \gamma^*\omega^{m-1} = 0.$$

Next given a lifted curve $\sigma(t) = (x(t), u(t))$ with $u(t) = x'(t)$ which lies in the indicatrix in each fiber, then

$$x'(t)dt = \sigma^* dx = \sigma^*\omega^0 e_0 + \sum_{i=1}^{m-1} \sigma^*\omega^i e_i ,$$

hence choosing $e_0 = x'(t)$, which is the unit vector in the direction of the position vector of the indicatrices, results in the conditions

$$\sigma^*\omega^0 = dt, \quad \sigma^*\omega^1 = 0, \ldots, \sigma^*\omega^{m-1} = 0.$$

Thus given $v_x \in S(M)$, it will have the form $v_x = \sigma_* \frac{\partial}{\partial t}$, and hence

$$< \sigma_* \frac{\partial}{\partial t}, \omega^0 >=< \frac{\partial}{\partial t}, \sigma^* \omega^0 >=< \frac{\partial}{\partial t}, dt >= 1.$$

In both cases e_0 has been chosen to be the position vector of the hypersurfaces in each fiber. Next let us change notation to reflect this choice by introducing 1-forms ϕ and $\eta^1, \ldots, \eta^{m-1}$ defined by the equation

$$dx = \phi e_0 + \sum_{\alpha=1}^{m-1} \eta^\alpha e_\alpha.$$

Then setting $I = \{\eta^1, \ldots, \eta^{m-1}\}$ we see that $\phi + I$ gives the affine system characterizing either of the two geometries, hence any family of coframings so defined simultaneously encodes centro-affine geometry and either the control or the Finsler geometry.

Next we extend the choice of m-frames to the principal $SO(m-1, \mathbf{R})$ bundle over the image of V in $T(\mathbf{R}^m)$ where the group $SO(m-1, \mathbf{R})$ is the stabilizer of the Blaschke metric normalized as in section 2. Thus we have frames

$$(e_0(x, u), e_1(x, u, S), \ldots, e_{m-1}(x, u, S))$$

with $x \in \mathbf{R}^m$, $u \in \mathbf{R}^{m-1}$, $S \in SO(m-1, \mathbf{R})$. As above these have dual 1-forms

$$(\phi(x, u), \eta^1(x, u, S), \ldots, \eta^{m-1}(x, u, S))$$

defined by $dx = \phi e_0 + \sum \eta^\alpha e_\alpha$.

Utilizing the results of section 2, we now further adapt the framing and coframing so that on each fiber we have complementary forms $(\mu^1, \ldots, \mu^{m-1})$ satisfying

$$d_{fiber} \begin{pmatrix} e_0 \\ e \end{pmatrix} = \begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi + \Delta \end{pmatrix} \begin{pmatrix} e_0 \\ e \end{pmatrix},$$

and such that the exterior derivatives restricted to the fibers satisfy

$$d_{fiber}(\phi, \eta) = (\phi, \eta) \wedge \begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi + \Delta \end{pmatrix}.$$

Since only the fiber geometry is specified the matrix

$$\begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi + \Delta \end{pmatrix}$$

is only determined mod base, and as a consequence the full exterior derivative has the form

$$d(\phi,\eta) = (\phi,\eta)\wedge \begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi+\Delta \end{pmatrix} \quad + \quad \text{terms quadratic in the base.}$$

We note that the terms quadratic in the base are precisely the terms quadratic in ϕ, η. Analytically this means that every extension has the form

$$d\phi = -\eta\wedge {}^t\mu + \phi\wedge\eta a + \eta\wedge A^t\eta$$

where a is a vector and A is a matrix. If however, we change such an extended coframe by letting

$${}^t\bar{\mu} = {}^t\mu + a\phi - A^t\eta,$$

then we have

$$d\phi = -\eta\wedge {}^t\bar{\mu}.$$

This equation does not define $\bar{\mu}$ uniquely, since by Cartan's lemma there is an arbitrariness of the form

$${}^t\tilde{\mu} = {}^t\bar{\mu} + C^t\eta \quad \text{where} \quad {}^tC = C.$$

Next we rewrite the analogous equation for $d\eta$, but replacing μ by $\bar{\mu}$ to get

$$d\eta = \phi\wedge\bar{\mu} + \eta\wedge(\Phi+\Delta) + \phi\wedge\eta B + \eta\wedge T^t\eta,$$

where B is a matrix and T is a 3-tensor. We note that the last two terms include the correction for replacing μ by $\bar{\mu}$ not only in the first term , but also in Δ which was linear in the μ.

The standard algebraic lemma used in the fundamental theorem of Riemannian geometry shows that we can write the $\eta\wedge T^t\eta$ term as $\eta\wedge\tau$, where τ is a skew-symmetric matrix of 1-forms in the η's. Finally we split the matrix B into its symmetric and skew-symmetric parts

$$B = \underbrace{B_1}_{\text{skew}} + \underbrace{B_2}_{\text{symmetric}}$$

With this preparation we can now modify $\bar{\mu}$ and Φ by

$$\tilde{\mu} = \bar{\mu} + \eta\, B_2 \quad \text{and} \quad \widetilde{\Phi} = \Phi - B_1\phi + \tau$$

to get

$$d\eta = \phi\wedge\tilde{\mu} + \eta\wedge(\widetilde{\Phi}+\Delta).$$

Theorem 5:. *The structure equations*

$$\begin{aligned} d\phi &= -\eta \wedge {}^t\tilde{\mu} \\ d\eta &= \phi \wedge \tilde{\mu} + \eta \wedge (\widetilde{\Phi} + \Delta) \\ {}^t\widetilde{\Phi} &= -\widetilde{\Phi}, {}^t\Delta = \Delta \end{aligned}$$

uniquely determine all the forms $\tilde{\mu}$*,* Δ *and* $\widetilde{\Phi}$*.*

Proof. The arbitrariness in $\tilde{\mu}$ was used up in the last normalization, thus making it unique. Δ is a cubic form which is unique once the $\tilde{\mu}$ are unique. $\widetilde{\Phi}$ is then unique by the same argument that proves the uniqueness of the Levi-Civita connection in Riemannian geometry. □

These 1-forms ϕ, η and $\tilde{\mu}$ now define an e-structure and hence a solution of the equivalence problem see [Ga89]. If we differentiate these structure equations and drop all the tildes we obtain the full set of structure equations

$$\begin{aligned} d\phi &= -\eta \wedge {}^t\mu \\ d\eta &= \phi \wedge \mu + \eta \wedge (\Phi + \Delta) \\ d\mu &= -\Phi \wedge {}^t\mu + \eta P^t\mu + \phi Q^t\eta + \eta R^t\eta, \end{aligned}$$

where P is a tensor, Q is a matrix and R is a tensor, Φ is skew-symmetric and Δ is a cubic form. Differentiating once more gives the Bianchi identities which show that P, Q, R have the symmetries of the Finsler geometry associated to the general variational problem

$$\delta \int \phi = 0.$$

There are no constraints since the equivalence calculation makes ϕ into the Cartan or Hilbert form and by a Hamilton's principle see [Ga89] and [GaShWi89], the constrained problem

$$\delta \int_{\eta=0} \phi = 0,$$

is equivalent to the unconstrained problem

$$\delta \int \phi = 0.$$

Theorem 5 gives a solution to the equivalence problem for Finsler structures. From the structure equations, we clearly see two important connections for the structure. One is the torsion free connection[2] from Chern's 1948 paper [Ch48]

$$\begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi + \Delta \end{pmatrix}$$

and the other is Cartan's metric compatible connection [Ca33, Ca34]

$$\begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi \end{pmatrix} \text{ with torsion } \begin{pmatrix} 0 \\ \Delta \end{pmatrix}.$$

Of course if the Blaschke metric is not negative definite, but still non-degenerate, the analysis is similar with the usual pluses and minuses taken into account.

7. An open question

In section two we developed the centro-affine geometry of hypersurfaces and found two tensors that characterize surfaces up to a centro-affine motion: the centro-affine metric II_{CA} and the centro-affine Pick form P_{CA}. In section four we showed that the natural notion of equivalence of Finsler structures induces a centro-affine action on the fibers of the sphere bundle. To make sense of this we view each indicatrix as a convex hypersurface of a tangent space. We continue by adapting centro-affine coframes to each indicatrix and show that there is a unique choice of coframes that extends the centro-affine structure equations in each fiber to hold over the entire manifold. In this way we construct Chern's connection for Finsler structures. Under this correspondence, the Minkowski potential H_{abc} found in Chern's connection exactly corresponds to the centro-affine Pick form P_{CA}, which we denoted by $C^{\alpha}_{\beta\gamma}$.

The above correspondence shows that the Minkowski potential is really an invariant of the centro-affine structure of the indicatrix. This brings up a rather open ended question. What centro-affine properties of the indicatrix can we determine from the Pick form and what do these properties mean for the Finsler structure?

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