FINSLER GEOMETRY IN LOW DIMENSIONAL CONTROL THEORY

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Abstract. Following an introduction to control theory, we show how Finsler geometries occur in certain classes of control systems.

1. Introduction

In this paper we will study the problem of feedback equivalence of control systems. We will identify the major geometric structures and we will see in two examples that the local properties of the control systems are identical to those of a Finsler metric. We will exhibit a simple geometric condition for checking when a control problem does in fact come from a Finsler metric. In that case, we will see that the natural time optimal control problem has solutions with closed loop controls, and that these solutions are exactly the geodesics of the Finsler metric. We also mention briefly the role of centro-affine geometry in feedback control and how it relates to the fact that the vanishing of the torsion of a Finsler metric implies that it is Riemannian. A detailed discussion of the centro-affine geometry of hypersurfaces is given in [4], and an application of it to control theory can be found in [5].

2. Control Theory

2.1. **Control systems.** Consider the example of the kinematic car moving on the xy-plane, shown in figure 1. We specify the position, or state, of the car with the four variables (x, y, θ, φ). The pair (x, y) gives the coordinates of the center of the rear axle on the xy-plane. The variable θ is the angle the car makes relative to the horizontal, and the variable φ is the angle the front wheels make relative to the car.

We will assume that the wheels do not slip as the car moves on the plane. We model these conditions by requiring for each axle that the sideways velocity of its midpoint be zero. These conditions are thus

$$\big(dx, dy\big) \cdot \big(-\sin\theta, \cos\theta\big) = 0$$
$$\big(d(x + L\cos\theta), d(y + L\sin\theta)\big) \cdot \big(-\sin(\theta+\varphi), \cos(\theta+\varphi)\big) = 0$$

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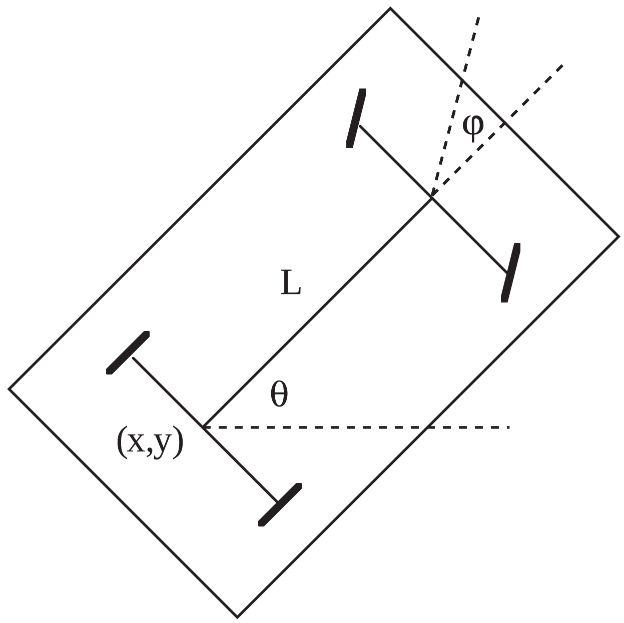

Figure 1. Kinematic Car

which simplify to

$$-\sin\theta\,dx + \cos\theta\,dy = 0$$
$$L\cos\varphi\,d\theta - \sin\varphi\big[\cos\theta\,dx + \sin\theta\,dy\big] = 0.$$

These equations show that the velocity vector for the car can be written in the following form

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\varphi}\end{pmatrix} = u^1\begin{pmatrix}\cos\theta\\ \sin\theta\\ L^{-1}\tan\varphi\\ 0\end{pmatrix} + u^2\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{2.1}$$

We see that u^1 represents the speed of the point (x, y) and u^2 represents the speed at which the front wheels turn.

The variables u^1 and u^2 play a special role in equation (2.1). If we substitute for each u^i a smooth function $u^i = u^i(t)$, then equation (2.1) becomes a determined system of ordinary differential equations. This system will have a unique integral curve for each initial condition. We see that the functions $(u^1(t), u^2(t))$ determine state space trajectories for the car. As such, we view the variables (u^1, u^2) as *controls* for the motion of the kinematic car. This example motivates the following local definition of a control system.

Definition 2.1. Let X be an open subset of $\mathbb{R}^n$ and let U be an open subset of $\mathbb{R}^m$, $m < n$. A *control system* on $X \times U$ is an underdetermined system of ordinary differential equations,

$$\frac{dx}{dt} = f(x,u), \quad x \in X, u \in U, \tag{2.2}$$

where $f\colon X \times U \to \mathbb{R}^n$ is a smooth function. X is called the *state space* of the control system and U is called the *control space*. The variables $x^1, \ldots, x^n$ are called *the states* and the variables $u^1, \ldots, u^m$ are called *the controls*.

Notice that substituting a smooth U-valued function $u = u(t,x)$ into equation (2.2) makes it a determined system. Specifying an initial state x_0 for an initial time t_0 will then determine a unique state space trajectory for the control system. Usually one only considers control functions of two types. They are the *open loop controls*, $u = u(t)$, and the *closed loop controls*, $u = u(x)$. In practice, an open loop system usually indicates the presence of a human operator, while a closed loop system indicates an automatic process where the particular control function is predetermined off-line. Observe that with closed loop controls equation (2.2) will be autonomous.

Definition 2.1 gives a local presentation of a control system defined on a manifold. Although we will not be using the global definition of a control system, the interested reader will find one given in [2], as well as a theorem showing that control systems are realized as the pullback of a certain universal system.

We will be interested in two important questions from control theory. One is the question of optimal control of a system, the other is the question of feedback equivalence of a control system.

2.2. **Optimal control.** Let $x_0, x_1 \in X$. We will say that x_1 is *reachable* from x_0 if there is a control $u(t)$, $t_0 \le t \le t_1$, and a curve $x(t)$ defined on $t_0 \le t \le t_1$ satisfying equation (2.2) and the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$. The control $u(t)$ is said to join x_0 to x_1. Suppose we are given a cost functional defined on solutions curves to (2.2) by

$$\mathcal{F}(x,u) = \int_{t_0}^{t_1} F(x(t), u(t))\, dt. \tag{2.3}$$

An example of a cost functional for the kinematic car is

$$\mathcal{F}(x,u) = \frac{1}{2}\int_{t_0}^{t_1} \left(u^1\right)^2 + \left(u^2\right)^2 dt.$$

We will be concerned with the example where $F(x,u) \equiv 1$, which means that $\mathcal{F}(x,u)$ gives the time required to traverse the curve.

If x_1 is reachable from x_0, then we may try to solve the following problem. Among all controls joining x_0 to x_1, find those that optimize the cost functional (2.3). This is the optimal control problem. There is a clear similarity between this problem and the classical calculus of variations with constraints. It is fairly easy to show that every calculus of variations problem can be cast as an optimal control problem [8, page 225].

2.3. **Feedback equivalence.** There are several notions of equivalence for control systems. We will be working with what is currently call *static state feedback*. In earlier years this was called "state feedback" or simply "feedback". In recent years the term "feedback" sometimes refers to the more general notion of "dynamic feedback". We will use "feedback" to mean "static state feedback" rather than "dynamic feedback".

Definition 2.2. A *feedback transformation* is a locally defined diffeomorphism $(\bar{x}, \bar{u}) = \Phi(x, u)$ such that

$$\begin{aligned} \bar{x} &= \phi(x) \\ \bar{u} &= \psi(x, u), \end{aligned} \tag{2.4}$$

where $\phi(x)$ is a state space diffeomorphism.

We note that a feedback transformation is simply a diffeomorphism of $X \times U$ that preserves the fibering of $X \times U \to X$. The set of feedback transformations forms a pseudo-group, and we use this pseudo-group to define feedback equivalence of control systems.

Definition 2.3. Let

$$\frac{d\bar{x}}{dt} = \bar{f}(\bar{x}, \bar{u}) \tag{2.5}$$

be a control system on $\overline{X} \times \overline{U}$. The system (2.2) is *feedback equivalent* to (2.5) if there is a feedback transformation $\Phi : X \times U \to \overline{X} \times \overline{U}$ such that every integral curve $\big(x(t), u(t)\big)$ of (2.2) maps to an integral curve of (2.5).

Feedback equivalence can also be characterized in terms of Pfaffian systems. To get to these systems, we will introduce some more structures. Consider the following pull-back bundle:

$$\begin{array}{ccc} p^*TX & \xrightarrow{\hat{p}} & TX \\ {\scriptstyle\hat{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ X \times U & \xrightarrow{p} & X \end{array} \tag{2.6}$$

where $p : X \times U \to X$ is projection onto X. Definition 2.1 of a control system adds two important maps to this commutative diagram. The first is the map $V : X \times U \to TX$

$$V(x, u) = \sum_{k=1}^{n} f^k(x, u) \frac{\partial}{\partial x^k} \tag{2.7}$$

and the second map is a section of $\hat{\pi}$

$$e_0(x, u) = \sum_{k=1}^{n} f^k(x, u) \partial_k. \tag{2.8}$$

In equation (2.8) we use the notation $\partial_1, \ldots, \partial_n$ to denote the unique local framing of p^*TX that maps to the coordinate frame $\partial/\partial x^1, \ldots, \partial/\partial x^n$, i.e. $\hat{p}(\partial_i) = \partial/\partial x^i$. As a consequence we see that $V = \hat{p} \circ e_0$.

One thing to notice is that for each x, $V(x,\cdot)$ maps $U \to T_xX$. We will always assume that this map has maximal rank m for each x. This is equivalent to the condition that the rank of the $n \times m$ matrix $\partial f/\partial u$ equals m, and implies that the image of $V(x,\cdot)$ is a regular m-dimensional submanifold of T_xX. In terms of control theory, this condition simply means that all of the control are essential.

From the control system we can define the following Pfaffian system on $X \times U$. For each (x,u), let

$$I|_{(x,u)} = p^*\{\eta \in T_x^*X \mid f(x,u) \rfloor \eta = 0\}. \tag{2.9}$$

Whenever $f(x,u) \neq 0$, the dimension of $I|_{(x,u)}$ equals $n-1$. To guarantee constant rank, we will henceforth assume that for all (x,u), $f(x,u) \neq 0$. We can also define an affine translate of I,

$$J|_{(x,u)} = p^*\{\varphi \in T_x^*X \mid f(x,u) \rfloor \varphi = 1\}. \tag{2.10}$$

The importance of the affine system comes from the following proposition.

Proposition 2.4. *Let $\gamma(t) = (x(t), u(t))$ be a smooth curve in $X \times U$. Then $\gamma(t)$ is an integral curve of the control system* (2.2) *if and only if for every $\varphi \in J|_{\gamma(t)}$ we have $\dot\gamma(t) \rfloor \varphi = 1$.*

Proof. Since $f(\gamma(t)) \neq 0$, we can choose a basis $\{\phi, \eta^1, \ldots, \eta^{n-1}\}$ of $T^*_{x(t)}X$ such that $v = f(\gamma(t))$ is the unique solution to the linear nonhomogeneous equations

$$\begin{aligned} v \rfloor \phi &= 1 \\ v \rfloor \eta^i &= 0, \quad 1 \leq i \leq n-1. \end{aligned}$$

By construction, the affine space $J|_{\gamma(t)}$ consists of all 1-forms of the form $\varphi = p^*\phi + a_1\, p^*\eta^1 + \cdots + a_{n-1}\, p^*\eta^{n-1}$. Therefore $\dot\gamma(t) \rfloor \varphi = 1$ for all $\varphi \in J|_{\gamma(t)}$ if and only if

$$\begin{aligned} \dot\gamma(t) \rfloor p^*\phi &= p_*\dot\gamma(t) \rfloor \phi = \dot x(t) \rfloor \phi = 1 \\ \dot\gamma(t) \rfloor p^*\eta^i &= p_*\dot\gamma(t) \rfloor \eta^i = \dot x(t) \rfloor \eta^i = 0, \quad 1 \leq i \leq n-1, \end{aligned}$$

and these equations hold if and only if $\dot x(t) = f(\gamma(t))$. □

Corollary 2.5. *The integral curves of* (2.2) *are also integral curves of I. Moreover, every integral curve of I which does not annihilate J can be reparametrized to be an integral curve of* (2.2)*.*

Proof. The first part is clear. Let φ be any section of J, and observe that $J = \varphi + I$. If $\gamma(t)$ is an integral curve of I that does not annihilate J, then we may reparametrize γ so that $\dot\gamma \rfloor \varphi = 1$. Then $\dot\gamma \rfloor \hat\varphi = 1$ for all $\hat\varphi \in J$, and the proposition applies. □

This corollary shows that the integral curves of the control system are completely determined by its corresponding affine system, J. This is part of what we need to prove the following corollary.

Corollary 2.6. *Let J be the affine system on $X \times U$ corresponding to the control system* (2.2) *and let $\bar{J}$ be the affine system on $\overline{X} \times \overline{U}$ corresponding to the control system* (2.5)*. Then a diffeomorphism $\Phi : X \times U \to \overline{X} \times \overline{U}$ is a feedback equivalence if and only if $\Phi^* \bar{J} = J$.*

Proof. From the above remark we see that pulling $\bar{J}$ back to J is equivalent to mapping integral curves of (2.2) to integral curves of (2.5). The only thing left to show is that $\bar{x} = \phi(x, u)$ is independent of u. This follows immediately by observing that the span of $\bar{J}$ is $p^*(T^*\overline{X})$ and the span of J is $p^*(T^*X)$. □

This corollary shows how to express feedback equivalence as a G-structure equivalence. Let φ be any section of J and let $\{\eta^1, \ldots, \eta^{n-1}\}$ be $n-1$ independent sections of I. Using similar bared forms for sections of $\bar{J}$ and $\overline{I}$, we see that a diffeomorphism Φ is a feedback equivalence if and only if

$$\begin{aligned} \Phi^* \bar{\varphi} &= \varphi + a_1 \eta^1 + \cdots + a_{n-1} \eta^{n-1} \\ \Phi^* \bar{\eta}^\sigma &= \qquad b_1^\sigma \eta^1 + \cdots + b_{n-1}^\sigma \eta^{n-1}, \quad \sigma = 1, \ldots, n-1 \end{aligned} \tag{2.11}$$

It is also worth noting that relative to a set of J-adapted 1-forms

$$\{\varphi, \eta^1, \ldots, \eta^{n-1}\}$$

there is a dual frame field $\{e_0, e_1, \ldots, e_{n-1}\}$ consisting of sections of $p^*TX \to X \times U$. This frame field satisfies the equation

$$dp = \hat{p} \circ e_0 \, \varphi + \hat{p} \circ e_1 \, \eta^1 + \cdots + \hat{p} \circ e_{n-1} \, \eta^{n-1} \tag{2.12}$$

and e_0 is the canonical section in equation (2.8).

We can now show that to every control system we can associate a natural optimal control problem. The optimal control problem is the kind of variational problem studied in Griffiths's book [7]. Moreover, feedback equivalence of the control system is identical to the simple equivalence of the variational problem (as defined in [1]).

In Griffiths's book, a variational problem is determined by a Pfaffian system I and a 1-form ϕ mod I. The problem is then to find the extremals of the functional $\int \phi$, where the integral is taken over integral curves of I. One of the fundamental structures in this problem is the affine space $\phi + I$.

In the control setting, the affine space J can always be written as $\varphi + I$, where φ is well defined mod I. The integral curves of I correspond to the integral curves of the control system (2.2). So if we consider the problem of finding the extremals of the functional $\int \varphi$ over integral curves of the control system, this optimal control problem has the exact form of the variational problems studied by Griffiths.

This optimal control problem is quite natural. By proposition 2.4, if $\gamma(t)$ is an integral curve then $\gamma^* \varphi = dt$, so the integral

$$\int_\gamma \varphi = \int_{t_0}^{t_1} dt = t_1 - t_0$$

simply gives the total time taken to traverse the trajectory. Thus this is the time optimal control problem mentioned in section 2.2.

Finally, as discussed in [1], the simple equivalence problem for a variational problem consists of looking for diffeomorphisms that preserve the Pfaffian system I and preserve the 1-form ϕ mod I. This is clearly equivalent to preserving the affine system $\phi + I$. Applying this condition to the time optimal variational problem, we see that the simple equivalence of the variational problem is identical to the feedback equivalence of the control system.

3. Examples

3.1. **Two states and one control.** We will consider the lowest dimensional example, where $X \subset \mathbb{R}^2$ and $U \subset \mathbb{R}$. A detailed study of the feedback equivalence of this system using Cartan's method of equivalence can be found in [6]. Studying this example gave us the first indication that a certain class of control systems has a feedback invariant Finsler metric associated to it in a natural way.

We will not go through the detailed calculations of the feedback equivalence problem for this system, but we will outline the broad steps (see [6] for full details). An adapted coframe for the control system consists of three independent 1-forms, $\{\varphi_0, \eta_0, \mu_0\}$, such that the span of $\{\varphi_0, \eta_0\}$ equals the span of $\{dx^1, dx^2\}$, $f(x,u) \rfloor \varphi_0 = 1$ and $f(x,u) \rfloor \eta_0 = 0$. From equations (2.11) we see that the group of the G-structure is the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ c_1 & c_2 & c_3 \end{pmatrix} \qquad b \cdot c_3 \neq 0$$

and the canonical form on the G-structure of adapted coframes is

$$\begin{pmatrix} \varphi \\ \eta \\ \mu \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \eta_0 \\ \mu_0 \end{pmatrix}. \tag{3.1}$$

As we proceed with the calculation, we will give the meaning of invariants in terms of the submanifold of TX determined by the image of the function $V(x,u)$ defined in equation (2.7). Note in particular that if we fix x and let u vary, then the image of V is a curve in the two dimensional vector space T_xX which is parametrized by $f(x,u)$. The invariants we compute will describe properties of this family of curves.

From equation (3.1) we see that the exterior equations

$$\begin{aligned} d\varphi &\equiv A\,\varphi \wedge \mu \mod \eta \\ d\eta &\equiv B\,\varphi \wedge \mu \mod \eta \end{aligned}$$

hold. The invariant B will vanish exactly when $f(x,u) \wedge \partial f / \partial u(x,u) = 0$, which means that the position vector $f(x,u)$ is tangent to the curve. If B vanishes identically, then a short calculation shows that we can transform

$f(x,u)$ to $u\binom{1}{0}$ by a feedback transformation. This case does not lead to a Finsler metric.

Assuming that $B \neq 0$, a calculation shows that we may restrict to the $B = 1$ submanifold. Since $d\big(\varphi - A\,\eta\big) \equiv 0 \mod \eta$ we see that we may also restrict to the $A = 0$ submanifold. These restrictions lead to the equation

$$d\varphi = C\,\varphi \wedge \eta + D\,\mu \wedge \eta.$$

The invariant D vanishes exactly when $\partial f/\partial u \wedge \partial^2 f/\partial u^2 = 0$, in other words at inflection points of the curve. If D vanishes identically, then the image of $f(x,u)$ is an affine line in each tangent space, and a short calculation shows that we can transform it to $\binom{0}{1} + \binom{1}{0}u$. We do not get a Finsler metric in this case, either.

Assuming that $D \neq 0$, a calculation shows that it is acted on by a square, so we may restrict to $D = \varepsilon = \pm 1$. The sign of D is an invariant indicating the convexity of the curve relative to the origin. The curve is convex for $\varepsilon = 1$ and concave for $\varepsilon = -1$. Seeing that we can write $d\varphi = \varepsilon(\mu + \varepsilon C\varphi) \wedge \eta$ shows that we can restrict to $C = 0$. This simplifies the equation for $d\varphi$ to $d\varphi = \varepsilon\,\mu \wedge \eta$.

After the restrictions $A = 0$, $B = 1$ the equation for $d\eta$ has the form

$$d\eta = -\mu \wedge \varphi + H\,\varphi \wedge \eta + I\,\eta \wedge \mu.$$

Seeing that we can write $d\eta = -(\mu + H\,\eta) \wedge \varphi + I\,\mu \wedge \eta$ shows that we may restrict to $H = 0$. The five normalizations — $A = 0$, $B = 1$, $C = 0$, $D = \varepsilon$ and $H = 0$ — determine the five parameters of the structure group and thus pick a unique feedback invariant coframe of $X \times U$. The structure equations for this coframe are

$$\begin{aligned} d\varphi &= \varepsilon\,\mu \wedge \eta \\ d\eta &= -\mu \wedge \varphi + I\,\eta \wedge \mu \\ d\mu &= -K\,\varphi \wedge \eta + J\,\eta \wedge \mu, \end{aligned} \tag{3.2}$$

where the form of the equation for $d\mu$ comes from $d^2\varphi = 0$. A consequence of $d^2\eta = 0$ is that the invariant J is the derivative of I in the φ direction.

It was these structure equations that first alerted us to the possibility that there may be an intrinsic Finsler metric associated to the control system. We observed that these equations were identical, up to a plus or minus sign on the invariants, to structure equations (I) Cartan obtained in his generalized metric spaces paper [3, page 120]. The questions we needed to answer were where is the metric and when is it actually Finsler.

It turns out that the metric is located on p^*TX defined in equation (2.6). Let $\{e_0, e_1\}$ be the sections of p^*TX dual to $\{\varphi, \eta\}$. For each $(x,u) \in X \times U$, these vectors form an orthonormal basis for the (possibly pseudo-) metric $(\varphi)^2 + \varepsilon\,(\eta)^2$ on T_xX. From this construction we see that $\hat{p} \circ e_0(x,u) = V(x,u)$, which simply traces out the curve $f(x,u)$ in each tangent space T_xX, describes a U-parametrized family of unit vectors in each tangent space (see figure 2).

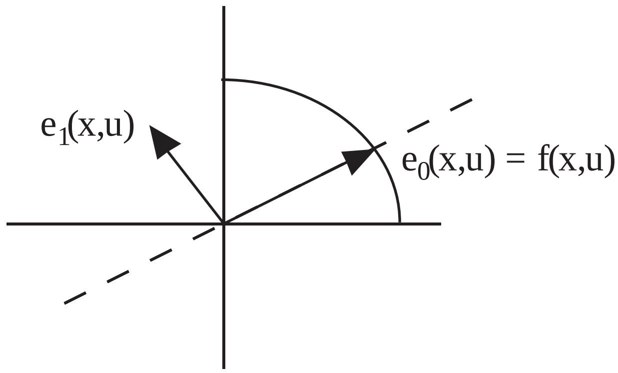

FIGURE 2. Orthonormal Frame

From the geometry of figure 2 we see that the curve traced out by $f(x,u)$ would have to be the indicatrix of the Finsler metric. This tells us how to determine when the metric is Finsler. For each $x \in X$, the curve traced by $f(x,u)$ in T_xX must be a strongly convex simple close curve which is centrally symmetric about the origin. In other words, $f(x,u)$ must be a parametrization of the indicatrix in T_xX.

Returning to the structure equations, we see that they define two connections; the metric compatible connection with torsion

$$d\begin{pmatrix}\varphi\\ \eta\end{pmatrix} = -\begin{pmatrix}0 & -\varepsilon\,\mu\\ \mu & 0\end{pmatrix}\wedge\begin{pmatrix}\varphi\\ \eta\end{pmatrix} + \begin{pmatrix}0\\ I\,\eta\wedge\mu\end{pmatrix} \tag{3.3}$$

and the torsion free connection

$$d\begin{pmatrix}\varphi\\ \eta\end{pmatrix} = -\begin{pmatrix}0 & -\varepsilon\,\mu\\ \mu & I\,\mu\end{pmatrix}\wedge\begin{pmatrix}\varphi\\ \eta\end{pmatrix}. \tag{3.4}$$

As usual, the vanishing of the torsion I in equation (3.3) shows that the metric $(\varphi)^2 + \varepsilon\,(\eta)^2$ drops to a (possibly pseudo-) Riemannian metric on X. An interesting way to get this result comes from considering connection (3.4) and the dual equations

$$\begin{aligned} de_0 &= \qquad\qquad e_1\,\mu \\ de_1 &= -\varepsilon\, e_0\,\mu + e_1\, I\,\mu \end{aligned} \tag{3.5}$$

The interesting feature of equations (3.5) is that they are also the centro-affine Frenet equations for a curve in $\mathbb{R}^2$. *Centro-affine geometry* is the study of the $GL(n,\mathbb{R})$ invariants of submanifolds of $\mathbb{R}^n$. When studying the $GL(2,\mathbb{R})$ invariants of curves in $\mathbb{R}^2$, we derive Frenet equations that look exactly like equations (3.5). In the Frenet equations e_0 is the position vector of the curve and e_1 is tangent. The form μ gives a $GL(2,\mathbb{R})$ invariant arclength, ds, on the curve. The value of ε determines the convexity of the curve relative to the origin and the invariant I is called the *centro-affine curvature* of the curve. The vanishing of the centro-affine curvature exactly characterizes the nondegenerate quadrics centered at the origin. Since a

Finsler metric is Riemannian exactly when its indicatrix is a nondegenerate quadric centered at the origin, it is clear why the vanishing of I characterizes Riemannian metrics. A more detailed discussion of centro-affine curves in $\mathbb{R}^2$ and its application to the feedback equivalence problem can be found in [9].

Return now to the time optimal control problem, $\int \varphi$. As we saw earlier, feedback equivalence of the control system is identical to simple equivalence of the variational problem. Thus, the invariant 1-form φ must be the Cartan form for this problem. We may therefore replace the constrained variational problem with the free variational problem. The equation for $d\varphi$ (3.2) gives the Euler-Lagrange system for the variational problem

$$\{\eta, \mu\}. \tag{3.6}$$

We can use this system to get a result in control theory.

Theorem 3.1. *Given a control system* (2.2) *with structure equations* (3.2) *then for any initial point x_0 and any initial direction $f(x_0, u_0)$ there exists a closed loop control $u = u(x)$, with $u(x_0) = u_0$, whose integral curve is an extremal for the time optimal variational problem.*

Proof. Since the Euler-Lagrange system (3.6) is completely integrable, we know there are functions $T(x,u) \neq 0$, $g(x,u)$ such that $\mu \equiv T\,dg \bmod \eta$. Recalling that $\{\varphi, \eta\} = \{dx^1, dx^2\}$, we see that $0 \neq \mu \wedge \varphi \wedge \eta = T\,\partial g/\partial u\, du \wedge \varphi \wedge \mu$ which implies that $\partial g/\partial u \neq 0$. Therefore we can solve the equation $g(x,u) = g(x_0, u_0)$ for $u = u(x)$, $u(x_0) = u_0$. Substituting $u(x)$ into the control system (2.2) gives an equation with a unique integral curve $\gamma(t) = \big(x(t), u(x(t))\big)$ satisfying $x(0) = x_0$. Since γ is an integral curve $\gamma^*\eta = 0$. We also have that $\gamma^* dg = 0$ since $g(x, u(x))$ is constant, which implies $\gamma^*\mu = 0$. Hence γ is an integral curve of the Euler-Lagrange system and is therefore an extremal. $\square$

Corollary 3.2. *If the control system* (2.2) *represents a Finsler metric, then the time optimal extremals of theorem 3.1 are geodesics of the Finsler metric.*

Proof. Since $\eta = 0$ and $\mu = 0$ the curve is a geodesic for the metric connection (3.3). $\square$

3.2. **Three states and two controls.** The example of three states and two controls is similar to that of two states and one control (see [10] for detailed calculations). We have $X \subset \mathbb{R}^3$ and $U \subset \mathbb{R}^2$. The geometric condition identifying which control systems come from Finsler metrics is that for each x the surface traced out by $f(x,\cdot) : U \to T_xX$ be strongly convex and centrally symmetric. An J-adapted coframe consists of five independent 1-forms, $\{\varphi_0, \eta_0^1, \eta_0^2, \mu_0^1, \mu_0^2\}$, satisfying the conditions that $\{\varphi_0, \eta_0^1, \eta_0^2\}$ span $\{dx^1, dx^2, dx^3\}$, $f \rfloor \varphi_0 = 1$, $f \rfloor \eta_0^1 = 0$ and $f \rfloor \eta_0^2 = 0$. The canonical form on the G-structure of J-adapted coframes is

$$\begin{pmatrix}\varphi\\ \eta^1\\ \eta^2\\ \mu^1\\ \mu^2\end{pmatrix} = \begin{pmatrix}1 & a_1 & a_2 & 0 & 0\\ 0 & b_1^1 & b_2^1 & 0 & 0\\ 0 & b_1^2 & b_2^2 & 0 & 0\\ c_1 & v_1^1 & v_2^1 & w_1^1 & w_2^1\\ c_2 & v_1^2 & v_2^2 & w_1^2 & w_2^2\end{pmatrix}\begin{pmatrix}\varphi_0\\ \eta_0^1\\ \eta_0^2\\ \mu_0^1\\ \mu_0^2\end{pmatrix}$$

or in a more abbreviated form

$$\begin{pmatrix}\varphi\\ \eta\\ \mu\end{pmatrix} = \begin{pmatrix}1 & a & 0\\ 0 & B & 0\\ c & V & W\end{pmatrix}\begin{pmatrix}\varphi_0\\ \eta_0\\ \mu_0\end{pmatrix}. \tag{3.7}$$

From this equation we can see that

$$d\varphi \equiv \varphi\wedge m\,\mu \mod \eta$$
$$d\eta \equiv \varphi\wedge M\,\mu \mod \eta$$

where m is a vector and M is a 2×2 matrix of functions. The matrix M is nonsingular exactly when the position vector $f(x,u)$ is transverse to its tangent plane in T_xX. We will assume this is the case. Then we can set M equal to the 2×2 identity matrix. Having done this we see that $d(\varphi - m\cdot\eta) \equiv 0 \mod \eta$, which means we can set $m = 0$. In terms of the dual frames $\{e_0, e_1, e_2\}$, these normalizations mean that $de_0 \equiv e_1\,\mu^1 + e_2\,\mu^2 \mod \eta$, which means that e_1, e_2 are tangent to the surface.

The equation for $d\varphi$ is now

$$d\varphi = -{}^t\mu\wedge H\eta + \varphi\wedge k\eta + {}^t\eta\wedge K\,\eta, \tag{3.8}$$

where $H = (h_{\alpha\beta})$ is a symmetric matrix, k is a row vector and K is a skew-symmetric matrix. The symmetric form

$$\sum_{\alpha,\beta=1}^{2} h_{\alpha\beta}\,\mu^\alpha\mu^\beta$$

is an invariant form corresponding to the centro-affine second fundamental form. It is called a second fundamental form because

$$de_\alpha \equiv e_0\sum_{\beta=1}^{2} h_{\alpha\beta}\,\mu^\beta \mod e_1, e_2, \varphi, \eta.$$

H will be negative definite when the surface in T_xX is convex about the origin. Since this is required in order to be a Finsler metric, we will now assume that H is negative definite. We may diagonalize the form by choosing e_1, e_2 orthonormal. This gives $H = -I$. An easy calculation shows we can set $k = 0$ and $K = 0$, leaving $d\varphi = -{}^t\mu\wedge\eta$.

We can write the equation $d\eta$ as

$$d\eta = \varphi\wedge\mu + \Phi\wedge\eta + \Delta\wedge\eta,$$

where Φ is a skew-symmetric matrix of 1-forms and Δ is a symmetric matrix of 1-forms. With a careful choice of μ and Φ we will have that $\Delta^\alpha_\beta =$

$\sum_{\gamma=1}^{2} P^{\alpha}_{\beta\gamma}\,\mu^{\gamma}$, where $P^{\alpha}_{\beta\gamma}$ is symmetric in all three indices. Giving Δ this form determines the coframe. The symmetric 3-form $\sum P^{\alpha}_{\beta\gamma}\,\mu^{\alpha}\mu^{\beta}\mu^{\gamma}$ is called the *centro-affine Pick form.* The vanishing of this form exactly characterizes the nondegenerate quadric surfaces.

As in the earlier case we now have the metric compatible connection with torsion

$$d\begin{pmatrix}\varphi\\ \eta\end{pmatrix} = -\begin{pmatrix}0 & -{}^t\mu\\ \mu & \Phi\end{pmatrix}\wedge\begin{pmatrix}\varphi\\ \eta\end{pmatrix} + \begin{pmatrix}0\\ \Delta\end{pmatrix}.$$

Also as before, the vanishing of the Pick form identifies the Riemannian metrics.

The theorems for three states and two controls corresponding to 3.1 and 3.2 hold and the proofs are essentially the same. In fact, these examples generalize to the general case of n states and $n-1$ controls.

3.3. **An open problem.** This section discusses a problem relating to applying Cartan's method of equivalence to a problem in geometric control theory. We remarked earlier that the connection (3.4) exactly gives the centro-affine Frenet equations (3.5) to the invariant moving frame e_0, e_1. We view the control system $f(x,u)$ as determining a u-parametrized curve in each two dimensional tangent plane T_x. The restriction of the 1-form μ to a curve gives a centro-affine invariant arc-length. Relative to this invariant parameter, the frame e_0, e_1 represents the position vector and the velocity vector of the curve.

While it is interesting to note that this fiber geometry appears after we solve the feedback equivalence problem for scalar control systems in the plane, it is even more interesting to see how we can reverse the process and use the centro-affine Frenet equations to solve the feedback equivalence problem. The companion paper to this one, also in this volume and by Gardner and Wilkens, shows how to reverse the process for the general case of n-states and $(n-1)$-controls. The procedure allows one to very quickly solve the feedback equivalence problem for these control systems using the easily derived structure equations of centro-affine hypersurfaces. We are thus able to bypass the usual equivalence calculation, and we have geometric interpretations of the invariants in terms of centro-affine geometry.

With the successful application of centro-affine hypersurface theory to control theory, we sought a similar application of centro-affine curve theory. The general centro-affine Frenet equations for a curve in $\mathbb{R}^n$ were worked out in [4]. For example, a curve in $\mathbb{R}^5$ will have a Frenet frame e_0, e_1, e_2, e_3, e_4 with structure equations

$$\begin{aligned}
de_0 &= e_1\,\mu\\
de_1 &= e_1\,\kappa_1\,\mu + e_2\,\mu\\
de_2 &= e_1\,\kappa_2\,\mu + e_2\,2\kappa_1\,\mu + e_3\,\mu\\
de_3 &= e_1\,\kappa_3\,\mu + e_2\,3\kappa_2\,\mu + e_3\,3\kappa_1\,\mu + e_4\,\mu\\
de_4 &= e_0\,\mu + e_1\,\kappa_4\,\mu + e_2\,4\kappa_3\,\mu + e_3\,6\kappa_2\,\mu + e_4\,4\kappa_1\,\mu,
\end{aligned} \tag{3.9}$$

where $\mu = ds$ is the centro-affine arc-length, e_0 is the position vector, and $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are the centro-affine curvatures of the curve.

Suppose now that we have a control system (2.2) with $n = 5$ and $m = 1$. Then we may view $f(x, u)$ as defining a curve in each five dimensional tangent space T_x. We proceed exactly as we did in the case of hypersurfaces, and adapt a centro-affine Frenet frame to each of these curves. This gives a feedback invariant set of frames $e_0(x, u), \ldots, e_4(x, u)$ with $e_0(x, u) = f(x, u)$. This frame will satisfy equations (3.9) except that the equalities will be replaced by congruences modulo the dx^i's. The equations only define μ modulo the dx^i's as well. Dual to the frame is the set of 1-forms $\varphi, \eta^1, \eta^2, \eta^3, \eta^4$ defined by the equation

$$dx = e_0\,\varphi + e_1\,\eta^1 + e_2\,\eta^2 + e_3\,\eta^3 + e_4\,\eta^4. \tag{3.10}$$

This equation shows that the integral curves of the control system are exactly the curves $\gamma(t) = (x(t), u(t))$ for which $\gamma^*\eta^i = 0$, for $1 \le i \le 4$, and $\gamma^*\varphi = dt$. Therefore, up to parametrization, the integral curves of the control system are the integral curves of the Pfaffian system $I = \{\eta^1, \ldots, \eta^4\}$.

Continuing our attempt to mimic the case of hypersurfaces, we examine the structure equations for the dual 1-forms. Equations (3.9) imply the following partial equations

$$\begin{aligned}
d\varphi &\equiv \eta^4 \wedge \mu \\
d\eta^1 &\equiv \varphi \wedge \mu + \kappa_1\,\eta^1 \wedge \mu + \kappa_2\,\eta^2 \wedge \mu + \kappa_3\,\eta^3 \wedge \mu + \kappa_4\,\eta^4 \wedge \mu \\
d\eta^2 &\equiv \eta^1 \wedge \mu + 2\kappa_1\,\eta^2 \wedge \mu + 3\kappa_2\,\eta^3 \wedge \mu + 4\kappa_3\,\eta^4 \wedge \mu \\
d\eta^3 &\equiv \eta^2 \wedge \mu + 3\kappa_1\,\eta^3 \wedge \mu + 6\kappa_2\,\eta^4 \wedge \mu \\
d\eta^4 &\equiv \eta^3 \wedge \mu + 4\kappa_1\,\eta^4 \wedge \mu.
\end{aligned} \tag{3.11}$$

The congruences in equation (3.11) are taken modulo quadratic terms in $\varphi, \eta^1, \eta^2, \eta^3, \eta^4$.

Since the Pfaffian system I is a feedback invariant of the control system, the derived flag of I will also be an invariant. Equations (3.11) immediately show that the first derived system is $I^{(1)} = \{\eta^2, \eta^3, \eta^4\}$. Our equations show that η^2 can not be in the second derived system, but they do not show enough information about the derivatives of η^3 and η^4. All we can say for sure is that the derivatives have the form

$$\begin{aligned}
d\eta^3 &\equiv A\,\varphi \wedge \eta^1 \qquad (\text{mod } \eta^2, \eta^3, \eta^4) \\
d\eta^4 &\equiv B\,\varphi \wedge \eta^1 \qquad (\text{mod } \eta^2, \eta^3, \eta^4).
\end{aligned}$$

Here is the odd situation. If the function B does not equal 0, then η^4 can not be in $I^{(2)}$. This means that our special invariant coframe is not adapted to the derived flag. Unfortunately, there are examples of control systems with $B \neq 0$, so this can happen. Moreover, if we apply Cartan's equivalence method the problem, we end up with a coframe adapted to the derived flag. The unfortunate conclusion is that what worked so well for the case of hypersurfaces fails in the case of curves.

Here at last is the question. Why are these two cases so different? For hypersurfaces we have two approaches to solving the equivalence problem and they give the same result. For curves, the two approaches give different results. Why did this happen and is there a good way to reconcile the two approaches?

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