A geometrical approach to the motion planning problem for a submerged rigid body. ⋆

Ryan N. Smith [*1] **Monique Chyba** [**] **George R. Wilkens** [**] **Christopher J. Catone** [***]

[*] *Ocean & Resources Engineering Department, University of Hawaii at Manoa, Honolulu, HI, USA.*
[**] *Mathematics Department, University of Hawaii at Manoa, Honolulu, HI, USA;*
[***] *Department of Mathematics, Albright College, Reading, PA, USA;*

Abstract: The main focus of this paper is the motion planning problem for a deeply submerged rigid body. The equations of motion are formulated and presented by use of the framework of differential geometry and these equations incorporate external dissipative and restoring forces. We consider a kinematic reduction of the affine connection control system for the rigid body submerged in an ideal fluid, and present an extension of this reduction to the forced affine connection control system for the rigid body submerged in a viscous fluid. The motion planning strategy is based on kinematic motions; the integral curves of rank one kinematic reductions. This method is of particular interest to autonomous underwater vehicles which can not directly control all six degrees of freedom (such as torpedo shaped AUVs) or in case of actuator failure (i.e., under-actuated scenario). A practical example is included to illustrate our technique.

Keywords: Motion planning, geometric control, submerged rigid body, kinematic motions.

1. INTRODUCTION

Motion planning is best known as a classical problem in robotics, but has applications to such areas as manufacturing, computer animation, medical surgery and pharmaceutical drug design. The motion planning problem can be stated as follows: Given a complex environment and an object capable of movement in n degrees of freedom, find a collision-free path from an initial configuration (position and orientation) to a different final configuration which also respects any constraints on the system. Motion planning techniques were initially developed to create mechanical systems with motion autonomy. Under the proper circumstances, we can use these autonomous systems to reduce expenses and eliminate human involvement. This is particularly important in areas of high risk such as undersea research and exploration.

In ocean research and exploration, we have reached a point where underwater mechanical systems are a necessity. We are always pushing to go further and deeper into the ocean depths, which increases the risks for manned vessels. Among the many technological advances in underwater mechanical systems, autonomous underwater vehicles (AUVs) are attracting recent research interests. These vehicles are excellent candidates upon which to implement solutions to the motion planning problem.

The ocean provides a complex environment for the AUV to operate within and submerged vehicles have the capability to move in six degrees-of-freedom (DOF). As long as the vehicle has the ability to directly control each of the six DOF, the motion planning problem is easily addressed, and trajectories can even be optimized with respect to a cost function such as time or energy consumption *(Chyba et al., 2008a,b)*. However, AUVs are mechanical systems which inevitably malfunction for one reason or another; batteries fail, actuators quit and electronics short out. Considering the loss of actuator(s) or specific vehicle design, if the vehicle does not have direct control on one or more DOF, we consider it to be *under-actuated*. In this scenario, the motion planning problem is more difficult, as some final configurations may not be realizable. In this paper we address the motion planning problem for an under-actuated AUV.

To begin, we derive the equations of motion for a rigid body submerged in a *real fluid* subject to external restoring and dissipative forces. We write these equations as a forced affine connection control system (FACCS) on a differentiable configuration manifold, $Q = \mathbb{R}^3 \times \mathrm{SO}(3) \cong \mathrm{SE}(3)$. In Sections 3.1 and 3.2, we present two simplified versions of the equations of motion by making some basic assumptions about the vehicle. See *(Bullo and Lewis, 2005a)* for a general reference on geometric control of mechanical systems.

With our equations of motion expressed in the context of the geometric control framework, we can then consider solutions to the motion planning problem using a geometric reduction procedure. The kinematic reduction is a first order control system on the configuration space Q whose controlled trajectories are also controlled trajectories of

⋆ Research funded in part by NSF grant DMS-0608583.

the second order dynamic system, possibly up to reparameterization. Of particular interest for the under-actuated scenario, we will consider kinematic reductions of rank one which are called decoupling vector fields.

For an affine connection control system (ACCS), *(Bullo and Lewis, 2005a)* provide definitions for a kinematic reduction and a decoupling vector field. However, these definitions do not take external forces into account. For an ideal fluid, neglecting any restoring forces, we can use these developments to calculate some solutions to the motion planning problem for AUVs. Unfortunately, these solutions cannot be implemented on a test-bed vehicle, since external forces play a major role in the dynamics of the vehicle. Currently, a characterization of a kinematic reduction for the FACCS which includes the external dissipative and restoring forces experienced by a real vehicle does not exist. Hence, we must look to extend the known theory in order to calculate solutions to the motion planning problem which can be implemented onto a real AUV.

In this paper, we calculate decoupling vector fields for an AUV submerged in an ideal fluid subject to no restoring forces. We then show that the dissipative force of viscous drag can be absorbed into the affine connection. This allows us to partially extend the notions of kinematic reduction and decoupling vector field to a real fluid and express this mechanical system as an ACCS using the new affine connection. With this extension, we calculate the decoupling vector fields for an AUV submerged in a real fluid subject to no restoring forces. Section 6 presents a control strategy for a torpedo-shaped vehicle performing a survey mission calculated by concatenating the integral curves of the calculated decoupling vector fields. We show the continuous controls which may be implemented onto a test-bed vehicle.

2. BACKGROUND AND MOTIVATION

The motivation for this study began with the introduction of decoupling vector fields presented in *(Bullo and Lynch, 2001)* and with the expansion of this work in *(Bullo and Lewis, 2005b)*. The first article presents the definitions and lays the foundation for the more detailed controllability results and motion planning techniques contained within the later reference as well as within the text of *(Bullo and Lewis, 2005a)*. In these references, as well as this paper, the equations of motion for a class of simple mechanical systems are derived by use of the techniques of differential geometry and are presented as a second-order (forced) affine connection control system on the configuration space. The geometric formulation and approach to the submerged rigid body problem not only allows for a kinematic reduction to a first-order system, but makes analysis of under-actuated systems quite simple from the motion planning perspective.

Controllability of these under-actuated systems, from a kinematic standpoint, is checked via the Lie bracket distribution, for which computations can be lengthy but are not difficult. For an under-actuated, controllable system, we decouple the trajectory planning between zero velocity states following the integral curves of decoupling vector fields. These vector fields are defined on the configuration space Q, and define time scalable trajectories which do not violate the under-actuated constraints on the system. This method for trajectory planning and design was first seen in *(Lynch et. al, 1998)* and *(Lynch et. al, 2000)* with application to a three degrees-of-freedom robot with a passive third joint.

The contribution of this paper, and novelty of this research, is to extend this trajectory design technique developed in the previously mentioned references to practical applications. In particular, with application to AUVs, the objective of this research is to provide implementable, open-loop control strategies which steer the AUV along a trajectory which is determined by the calculated decoupling vector fields. Many trajectories and control strategies have been calculated and implemented onto a test-bed AUV with experimental results matching well with theoretical predictions (see e.g., *(Smith, 2008)*, *(Chyba and Smith, 2008)* and *(Chyba et al., 2008a,b)*. We remark that in these experiments, the control strategies were implemented in open-loop with no feedback.

We focus our efforts on the design of open-loop controls, as their successful implementation can lead to enhanced accuracy with reduced control effort by the vehicle. Acting as a control in the feedforward path of the control system, the open-loop controls steer the vehicle using an understanding of the vehicle's dynamic response to each type of control input. The controls calculated here also incorporate any under-actuation of vehicle. Upon calculating a trajectory which can be performed by the under-actated system, a further area of research is the implementation of a feedback controller to track the prescribed trajectory.

Since open-loop controls are based on the *a priori* knowledge of the inputs to the system, they are not robust against unknown disturbances. To remedy this, AUVs are typically controlled using a feedback or adaptive control law. Such a control feeds back an estimated correction factor into the actuation signal. In this manner, the vehicle can compensate or adapt to changes in the environment or correct for inaccuracies present in the hydrodynamic model. A literature survey of feedback controllers applicable to AUVs can be found in *(Smith, 2008)*.

The implementation of a feedback control is useful as long as there is a way to determine or estimate the error along the chosen trajectory. For underwater applications, vehicle navigation and positioning still present large problems to be addressed in this area. To this end, a hybrid system between open and closed-loop controls would provide a robust controller which utilizes the benefits of each.

Choosing a method of control for an underwater vehicle is highly dependent on the specific applications and tasks that the vehicle is designed to perform. Most AUVs in operation today use a combination or hybridization of control methods to provide the appropriate robustness and performance characteristics. For example, *(Tsukamoto et al., 1999)* experimented with the combination of an on-line neural-net controller, off-line neural-net controller, a fuzzy controller and a non-regressor based adaptive controller for position and velocity control of their system. Hybrid controllers such as this can provide good experimental results, however they share the characteristic that they all are model-free. This means that vehicle parameters are either estimated or learned by the system and the controllers need to be adjusted and/or tuned for the specific vehicle and application. For some vehicles, this tuning is implemented at the beginning of each mission.

With so many different vehicles in operation, model-free controllers have become wide-spread in AUV applications. There is no need for a precise hydrodynamic model, and these controllers provide robustness with respect to environmental disturbances. However, there is still need for improvement.

Improvement may come by reverting back to the original motion planning question; how does one calculate the controls which steer a vehicle from one configuration to another? To answer this question, we need to examine the mechanical system at hand. However, many current models do not entirely describe a submerged rigid body. For this we turn to a differential geometric architecture and utilize the equations of motion presented the following section. With this formulation, we are able to exploit symmetries, geometry and inherent non-linearities of an underwater vehicle to produce control strategies based on the model. Also, the coordinate invariant formulation on a differentiable manifold allows us to consider the intrinsic structure of the problem rather than a pseudo-structure attached through the choice of a specific coordinate system.

Some of the language and notation contained within this paper may not be common knowledge to the general control theory community, thus, we include a few definitions and remarks about the geometric tools used here. For a complete reference on geometric control theory, please see *(Bullo and Lewis, 2005a)* and the references contained therein. Additionally, for those interested in further motivation supporting the use of differential geometry for modeling mechanical systems, we refer you to *(Lewis, 2007)*.

Control strategies have been calculated by use of the previously mentioned method for a rigid body submerged in an ideal fluid without consideration of external forces or moments such as viscous drag and restoration from gravity and buoyancy. Such a mechanical system can be expressed by an ACCS. An ACCS is a 4-tuple $(Q, \nabla, \mathcal{Y}, U)$, where Q is the configuration manifold for the system, ∇ is an affine connection defined on Q, $\mathcal{Y}$ is a set of vector fields defined on Q and $U \subset \mathbb{R}^m$. We refer to the set $\mathcal{Y}$ as the set of input control vector fields.

In the section to follow, we present the equations of motion for a rigid body submerged in a viscous fluid as a FACCS. We present a development of a force in the framework of differential geometry and include viscous drag and restoring forces and moments in our equations of motion. A FACCS is a 5-tuple $(Q, \nabla, F, \mathcal{Y}, U)$, where F is a vector force acting on the system. We refer to F as a drift vector field, as F defines the dynamics of the system in the absence of control inputs.

To design our motion, we will consider a kinematic reduction of the second-order ACCS describing a submerged rigid body. Of particular interest to the motion planning problem considered here are the kinematic reductions of rank one which are referred to as decoupling vector fields. These decoupling vector fields have the property that every reparameterized integral curve is a trajectory for the second-order ACCS. We refer to the integral curves of the decoupling vector fields as the kinematic motions of the ACCS. A decoupling vector field V is characterized by the fact that V and $\nabla_V V$ must both be sections of the vector bundle defined by the distribution of the the span of $\mathcal{Y}$. Our working definition of kinematic reduction and the notion of a decoupling vector field do not exist for a FACCS. This paper provides a first extension of the kinematic reduction and decoupling vector fields to a FACCS.

For a fully-actuated system, every vector field is decoupling, and we can concatenate the integral curves of decoupling vector fields (parameterized to begin and end with zero velocity) to steer the vehicle from an initial configuration $\eta_{init} \in Q$ to a final configuration $\eta_f \in Q$. For the under-actuated scenario, the trajectory design is a bit more difficult. Since we are unable to control all six degrees-of-freedom, the vehicle may not be able to realize a given final configuration. Hence, we must first examine the controllability of the system (i.e., which configurations are realizable by the system). We call an ACCS kinematically controllable if the system can reach any configuration $\eta_f \in Q$ from any starting configuration $\eta_{init} \in Q$ in finite time using only kinematic motions. We refer the reader to Definition 8.23 and the remarks following this definition in *(Bullo and Lewis, 2005a)* for a method to determine the kinematic controllability of an ACCS.

When considering the motion planning problem in the case of an autonomous underwater vehicle, under-actuation is of major concern for many reasons. First of all, the vehicle needs to be prepared to deal with actuator failure(s) resulting from any number of mechanical issues. Secondly, since AUVs are limited by the power carried on-board, it may be beneficial to operate in an under-actuated, but fully controllable condition in an effort to conserve energy. Additionally, early consideration of these path planning results may assist in vehicle design to implement effective redundancy. Such consideration at the design stage could also aide in the construction of a fully controllable but under-actuated vehicle for more cost-effective applications.

Since this geometric formulation is model based, any symmetry and inherent geometric structure is exploited in the trajectory construction. For these reasons, along with the pure mathematical beauty of the geometric mechanics involved, we consider extending this theory to design and implement control strategies onto a test-bed AUV.

The geometric control theory as presented in the aforementioned references of this section, with direct regard to the concepts of a kinematic reduction and decoupling vector fields, is not directly applicable for implementation onto a test-bed AUV. Current results have only been applied to drift-free affine connection control systems (i.e., no compensation for viscous drag or restoring forces and moments). In the case of an actual underwater vehicle, a *forced* affine connection control system describes the dynamics of a submerged rigid body. Hence, an extension of the current theory is necessary for real world applications. Our contribution contained within this paper is to present an extension of the current theory applied to a torpedo-shaped underwater vehicle. This extension demonstrates an approach to incorporate the viscous damping into the affine connection and a method to calculate open-loop control strategies for the under-actuated system.

3. EQUATIONS OF MOTION

We derive the equations of motion for a controlled rigid body immersed in an ideal fluid (air) and in a real fluid (water). By *real fluid*, we mean a fluid which is viscous and incompressible with rotational flow. Here, we consider water to be a viscous fluid (real fluid) in order to emphasize the inclusion of the dissipative terms in the equations of motion. This motivation comes from our desire to apply our results to the design of trajectories for test-bed underwater vehicles.

In what follows, we summarize the derivation of the equations of motion for a submerged rigid body through the use of the language and tools of differential geometry. These equations are equivalent to existing AUV models, such as those presented in *(Fossen, 1994)*. A detailed derivation of the equations presented here along with a parallel derivation of a classic AUV model can be found in *(Smith, 2008)*.

In the sequel, we identify the position and the orientation of a rigid body with an element of SE(3): (b, R). Here $b = (b_1, b_2, b_3)^t \in \mathbb{R}^3$ denotes the position vector of the body, and $R \in \mathrm{SO}(3)$ is a rotation matrix describing the orientation of the body. The translational and angular velocities in the body-fixed frame are denoted by $\nu = (\nu_1, \nu_2, \nu_3)^t$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3)^t$ respectively. Notice that our notation differs from the conventional notation used for marine vehicles. Usually the velocities in the body-fixed frame are denoted by (u, v, w) for translational motion and by (p, q, r) for rotational motion, and the spatial position is usually taken as (x, y, z) (e.g., *(SNAME, 1950)*). However, since this paper focuses on geometric control theory, the chosen notation will prove more efficient.

It follows that the kinematic equations for a rigid body are given by

$$\dot{b} = R\,\nu \tag{1}$$

$$\dot{R} = R\,\hat{\Omega} \tag{2}$$

where the operator $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by $\hat{y}\, z = y \times z$. The space $\mathfrak{so}(3)$ is the Lie algebra associated to the Lie group SO(3), and is the space of skew-symmetric 3×3 matrices (i.e., $\mathfrak{so}(3) = \{R \in \mathbb{R}^{3\times 3} | R^t = -R\}$).

To derive the dynamic equations of motion for a rigid body, we let p be the total translational momentum and π be the total angular momentum, in the inertial frame. Let P and Π be the respective quantities in the body-fixed frame. It follows that $\dot{p} = \sum_{i=1}^{k} f_i$, $\dot{\pi} = \sum_{i=1}^{k} (\hat{x}_i\, f_i) + \sum_{i=1}^{l} \tau_i$ where f_i (τ_i) are the external forces (torques), given in the inertial frame, and x_i is the vector from the origin of the inertial frame to the line of action of the force f_i.

To represent the equations of motion in the body-fixed frame, we differentiate the relations $p = R\,P$, $\pi = R\,\Pi + \hat{b}\,p$ to obtain

$$\dot{P} = \hat{P}\,\Omega + E_F \tag{3}$$

$$\dot{\Pi} = \hat{\Pi}\,\Omega + \hat{P}\,\nu + \sum_{i=1}^{k} (R^t\,(x_i - b)) \times R^t\, f_i + E_T \tag{4}$$

where $E_F = R^t \left(\sum_{i=1}^{k} f_i\right)$ and $E_T = R^t \left(\sum_{i=1}^{l} \tau_i\right)$ represent the external forces and torques in the body-fixed frame respectively.

To obtain the equations of motion of a rigid body in terms of the linear and angular velocities, we calculate the total kinetic energy of the system:

$$T = \frac{1}{2} \begin{pmatrix} v \\ \Omega \end{pmatrix}^t \begin{pmatrix} \mathbb{I}_{11} & \mathbb{I}_{12} \\ \mathbb{I}_{12}^t & \mathbb{I}_{22} \end{pmatrix} \begin{pmatrix} v \\ \Omega \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \mathbb{I}_{11} & \mathbb{I}_{12} \\ \mathbb{I}_{12}^t & \mathbb{I}_{22} \end{pmatrix} = \begin{pmatrix} mI_3 + M_f & -m\hat{r}_{C_G} + C_f^t \\ m\hat{r}_{C_G} + C_f & J_b + J_f \end{pmatrix}. \tag{6}$$

Here m is the mass of the rigid body, I_3 is the 3×3-identity matrix and r_{C_G} is a vector which denotes the location of the body's center of gravity (C_G) with respect to the origin of the body-fixed frame. $J_b = diag(J_b^{\Omega_1}, J_b^{\Omega_2}, J_b^{\Omega_3})$ is the body inertia matrix and $M_f = diag(M_f^{\nu_1}, M_f^{\nu_2}, M_f^{\nu_3})$, $J_f = diag(J_f^{\Omega_1}, J_f^{\Omega_2}, J_f^{\Omega_3})$ and C_f are respectively referred to as the added mass, the added mass moments of inertia and the added cross-terms. Equation (5) can also be written as $T = \frac{1}{2}(\nu^t \mathbb{I}_{11} \nu + 2\nu^t \mathbb{I}_{12} \Omega + \Omega^t \mathbb{I}_{22} \Omega)$. Using $P = \frac{\partial T}{\partial \nu}$ and $\Pi = \frac{\partial T}{\partial \Omega}$, we have:

$$\begin{pmatrix} P \\ \Pi \end{pmatrix} = \begin{pmatrix} mI_3 + M_f & -m\hat{r}_{C_G} + C_f^t \\ m\hat{r}_{C_G} + C_f & J_b + J_f \end{pmatrix} \begin{pmatrix} \nu \\ \Omega \end{pmatrix}. \tag{7}$$

The kinetic energy of a rigid body in an interconnected-mechanical system is represented by a positive-semidefinite $(0, 2)$-tensor field on the configuration space Q. The sum over all the tensor fields of all bodies included in the system is referred to as the *kinetic energy metric* for the system. In this paper, the mechanical system is composed of only one rigid body, the origin of the body fixed frame is located at C_G ($r_{C_G} = 0$) and the added cross terms C_f are zero by assuming three planes of symmetry. Then, the kinetic energy metric is the unique Riemannian metric on $Q = \mathbb{R}^3 \times \mathrm{SO}(3)$ given by

$$\mathbb{G} = \begin{pmatrix} M & 0 \\ 0 & J \end{pmatrix}, \tag{8}$$

where $M = mI_3 + M_f$ and $J = J_b + J_f$. In the sequel, we will use $m_i = m + M_f^{\nu_i}$ and $j_i = J_b^{\Omega_i} + J_f^{\Omega_i}$, for $i = 1, 2, 3$. Thus, $M = diag(m_1, m_2, m_3)$ and $J = diag(j_1, j_2, j_3)$. As with any Riemannian metric, associated to $\mathbb{G}$ is its *Levi-Civita connection*: the unique affine connection that is both symmetric and metric compatible. The Levi-Civita connection provides the appropriate notion of acceleration for a curve in the configuration space by guarenteeing that the acceleration is in fact a tangent vector field along a curve γ; the Levi-Civita connection can be studied in more depth in *(Bullo and Lewis, 2005a)*. An affine connection control system allows us to present a coordinate invariant formulation of the dynamic equations of motion for a submerged rigid body. In the case Explicitly, if $\gamma(t) = (b(t), R(t))$ is a curve in SE(3), and $\gamma'(t) = (\nu(t), \Omega(t))$ is its pseudo-velocity as given in Equations (18) and (19), the acceleration is given by

$$\nabla_{\gamma'}\gamma' = \begin{pmatrix} \dot{\nu} + M^{-1}(\Omega \times M\nu) \\ \dot{\Omega} + J^{-1}(\Omega \times J\Omega + \nu \times M\nu) \end{pmatrix}, \qquad (9)$$

where ∇ denotes the Levi-Civita connection and $\nabla_{\gamma'}\gamma'$ is the covariant derivative of γ' with respect to itself.

In order to consider rigid body motion in a viscous fluid, we must incorporate external interaction into the equations of motion. These interactions come in the form of control inputs, buoyancy gravity, pressure gradients and friction, among others. These are characterized as forces. Since this theory is expressed in the differential geometric framework, the notion of a force is not as obvious as when viewed from the Newtonian point of view.

Suppose that a rigid body is subjected to a Newtonian force ($\mathbf{f}$) or torque (τ). We allow a general external force F to depend on position ($(b, R) \in \mathbb{R}^3 \times \mathrm{SO}(3)$), on velocities ($(\dot{b}, \dot{R}) \in T_{(b,R)}(\mathbb{R}^3 \times \mathrm{SO}(3))$) and on time. Then, for each $v_{(b,R)} \in TQ$, we define an element $F_{\mathbf{f},\tau}(t, v_{(b,R)}) \in T^*_{(b,R)}Q$ which models the effects of $\mathbf{f}$ and τ. Hence, a $\mathcal{C}^r$-force on a manifold Q is is a map $F : \mathbb{R} \times TQ \mapsto T^*Q$ with the property that F is a locally integrally class $\mathcal{C}^r$ bundle map over id_Q. We call a force time-independant if this map has the property that $F(t, v_{(b,R)}) = F_0(v_{(b,R)})$, and call a force basic if $F(t, v_{(b,R)}) = F_0(b, R)$ for some $\mathcal{C}^r$-covector field F_0 on Q. Viewing a force from this perspective, Newton's Second Law can be viewed as $\mathbb{G}^\flat(\nabla_{\gamma'}\gamma') = \Sigma F$, where the left hand side of the equation represents mass times acceleration. This leads to the equivalent expression, $(\nabla_{\gamma'}\gamma') = \mathbb{G}^\#(\Sigma F)$, where the right hand side is now viewed as the sum of the forces and moments divided by mass.

Now, let $X_1, ..., X_6$ be the standard left-invariant basis for SE(3) and let $\pi^1, \ldots, \pi^6$ be it's dual basis. Then, we can express an external force acting on the system as a one form

$$F_{\mathbf{f},\tau}(t, v_{(b,R)}) = \sum_{i=1}^{6} F_i(t, b, R, \nu, \Omega)\pi^i, \qquad (10)$$

where $F_i = \langle F_{\mathbf{f},\tau}, \pi^i \rangle$ is the i^{th} component of the force, $i = 1, \ldots, 6$. By definition, this force is a function taking values in T^*M, while a geometric acceleration takes its values in TM. Applying $\mathbb{G}^\#$, the inverse of the kinetic energy metric $\mathbb{G}$, to $F_{\mathbf{f},\tau}$ yields a TM valued function. Multiplication by the matrix $\mathbb{G}^\#$ represents a vector bundle isomorphism that essentially means divide by mass; hence $\mathbb{G}^\#(F(\gamma'(t))) \in TM$ and can be compared to an acceleration as desired. For the mechanical system we are considering,

$$\mathbb{G}^\# = diag\left(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3}, \frac{1}{j_1}, \frac{1}{j_2}, \frac{1}{j_3}\right). \qquad (11)$$

Thus, in matrix form, the external accelerations become

$$\mathbb{G}^\#(F(\gamma'(t))) = \left(\frac{F_1}{m_1}, \frac{F_2}{m_2}, \frac{F_3}{m_3}, \frac{F_4}{j_1}, \frac{F_5}{j_2}, \frac{F_6}{j_3}\right). \qquad (12)$$

Now we have an object which represents, in the language of differential geometry, the external Newtonian forces. As mentioned before, these external forces can be used to model the drag, buoyancy, restorative and control forces that act upon the rigid body.

First, we consider the forces and moments which arise from a potential function. These restoring forces can be viewed as a force (torque) which acts to pull the rigid body back to its original position or orientation. Potential functions are commonly known to store energy to be turned into kinetic energy later. In particular, given a potential function $V \in \mathcal{C}^\infty(Q)$ on Q its *potential or restoring force* is the basic force given by $F(t, v_q) = -\operatorname{grad} V(q) = -G^\# dV(q)$ for $q \in Q$. In this paper, we concern ourselves with the restoring forces arising from buoyancy and gravity. These forces and moments are independent of time and velocity.

Let us fix a spatial (earth fixed) reference frame $\Sigma_{spatial} = (O_{spatial}, \{s_1, s_2, s_3\})$, where the s_i are orthogonal unit vectors and s_3 points in the direction of gravity. We denote by $r_G = (x_G, y_G, z_G)$ ($r_B = (x_B, y_B, z_B)$) the location of C_G (C_B, the center of buoyancy) with respect to $O_{spatial}$. Now, the restoring force from the acceleration due to gravity acts directly at C_G, and is given by the potential function $V_G(\gamma(t)) = W(Rr_B + b) \cdot s_3$ where $W = mg$ is the weight of the rigid body and $\cdot$ denotes the inner product. Similarly, the force arising from the buoyancy is the force exerted by the fluid on the submerged volume of the vessel and acts at C_B and is given by the potential function $V_B(\gamma(t)) = B(Rr_B + b) \cdot s_3$, where $B = \rho g \mathcal{V}$ and ρ is the fluid density, g is the acceleration due to gravity and $\mathcal{V}$ is the submerged volume of the body. These restoring forces and moments can be combined into one term denoted by $P(\gamma(t)) = -(dV_G + dV_B)$. Writing this in matrix form we can express the potential forces from gravity and buoyancy as accelerations by

$$G^\# P(\gamma'(t)) = \begin{bmatrix} -\frac{1}{m_1}(W - B)s\theta \\ \frac{1}{m_2}(W - B)c\theta s\phi \\ \frac{1}{m_3}(W - B)c\theta c\phi \\ \frac{1}{j_1}((y_G W - y_B B)c\theta c\phi - (z_G W - z_B B)c\theta s\phi) \\ -\frac{1}{j_2}((z_G W - z_B B)s\theta - (x_G W - x_B B)c\theta c\phi) \\ \frac{1}{j_3}((x_G W - x_B B)c\theta s\phi + (y_G W - y_B B)s\theta) \end{bmatrix}. \qquad (13)$$

Note that if $C_G \neq C_B$, the two opposing restoring forces will induce a torque, referred to as the righting moment, if the vehicle rotates. The righting arm GZ depends on the distance between C_G and C_B and the list angle ϕ as seen in Figure 1. On the other hand, if $C_G = C_B$, then the vehicle will experience no torque that opposes orientation displacements.

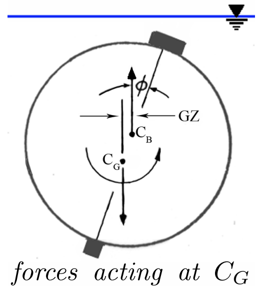

Fig. 1. *Potential forces acting at C_G and C_B and the righting arm for a submerged spherical vehicle.*

Not all external forces interacting with a submerged rigid body can be derived from a potential function. One example is the external force due to the viscosity of the surrounding fluid, or in our specific case, viscous drag. Such a force is called a dissipative force since it dissipates energy from the system.

Due to the shape and velocity of the test-bed vehicle under consideration, we assume that the dominant contribution of viscous drag comes from separation of the fluid from the body. This is commonly referred to as pressure (or form) drag. This force arises due to the pressure difference between the front and rear of the vehicle.

It is commonly known in hydrodynamics that for flows at high Reynolds number, or those characterized by flow separation, the drag force and moment is proportional to $\nu|\nu|$ and $\Omega|\Omega|$, respectively. For this research, we make the assumption that we have a drag force $D_\nu(\nu)$ and a drag momentum $D_\Omega(\Omega)$. We also assume non-coupled motion, implying that there are no off-diagonal terms. The contribution of these forces with respect to the velocities is given by

$$D_\nu(\nu_i) = \frac{1}{2} C_D \rho A_i \nu_i |\nu_i|, \; D_\Omega(\Omega_i) = \frac{1}{2} C_D \rho A_i \Omega_i |\Omega_i| \quad (14)$$

where C_D is the appropriate drag coefficient for the prescribed direction of motion [2], ρ is the density of the fluid, A_i is the projected surface area of the body in the direction of the velocity, ν_i are the translational velocities and Ω_i are the rotational velocities.

Remark 1. Note that in Equation (14), we express the drag force (moment) as proportional to $\nu|\nu|$ ($\Omega|\Omega|$). Since we consider motions in a single direction and assume steady flow, we can rewrite Equation (14) by use of the assumption $\nu|\nu| = \nu^2$ and $\Omega|\Omega| = \Omega^2$.

Since a vehicle may have different drag coefficients and different projected areas depending on the direction of the velocity, we define $F_i(\gamma'(t)) = D_i$ such that the total drag relation is given by $D_{total} = D_i |v_i| v_i$ for $i \in \{1, ..., 6\}$ and $v = (\nu_1, \nu_2, \nu_3, \Omega_1, \Omega_2, \Omega_3)$. Then, we let $\mathcal{D}_i = -\frac{D_i}{m_i}\nu_i^2$ for $i = 1, 2, 3$ and $\mathcal{D}_i = -\frac{D_i}{j_{i-3}}\Omega_{i-3}^2$ for $i = 4, 5, 6$.

Under these assumptions, the dissipative forces and moments (viscous drag forces and moments) depend on the square of the velocity of the body along a given trajectory γ. Hence, we can write $F_{drag}(\gamma'(t)) = \sum_{i=1}^6 F_i(\gamma'(t))\pi^i(\gamma'(t))$ which expands to

$$F_{drag}(\gamma'(t)) = \sum_{1}^{3} F_i(\gamma'(t))\nu_i^2 \pi^i + \sum_{i=1}^{3} F_{i+3}(\gamma'(t))\Omega_i^2 \pi^i. \quad (15)$$

Thus, we have the following expression

$$\begin{aligned} &\mathbb{G}^\#(F_{drag}(\gamma'(t))) \\ &= \sum_{i=1}^{3} \frac{F_i(\gamma'(t))}{\mathbb{G}_{ii}} \nu_i^2 X_i + \sum_{i=4}^{6} \frac{F_i(\gamma'(t))}{\mathbb{G}_{ii}} \Omega_{i-3}^2 X_i \\ &= \sum_{i=1}^{6} \mathcal{D}_i X_i, \end{aligned} \quad (16)$$

where $\mathbb{G}_{ij}$ is the i, j-entry of the kinetic energy matrix $\mathbb{G}$ as given before. The result is that the drag force is expressed as a geometric acceleration given by $\mathbb{G}^\#(F_{drag}(\gamma'(t)))$ which we can incorporate into the differential geometric equations of motion.

Remark 2. In practice, the drag forces and moments $F_i(\gamma'(t)) = D_i$ for $i = 1, ..., 6$ are estimated for the considered vehicle through model testing and estimation. In the sequel, we make use of pre-published drag force and moment estimations to calculate control strategies.

The final external forces which we consider are the input control forces. These are the forces with which we move the rigid body through the surrounding fluid. Throughout this paper, we assume that we have three forces acting at the center of gravity along the body-fixed axes and that we have three pure torques about these three axes. We will refer to these controls as the six DOF controls. We denote the controls by: $\sigma = (\varphi_\nu, \tau_\Omega) = (\varphi_{\nu_1}, \varphi_{\nu_2}, \varphi_{\nu_3}, \tau_{\Omega_1}, \tau_{\Omega_2}, \tau_{\Omega_3})$.

Remark 3. The above notion of control forces is not realistic from a practical point of view since underwater vehicle controls may represent the action of the vehicle's thrusters or actuators. The forces from these actuators generally do not act at the center of gravity and the torques are obtained from the momenta created by the forces. As a consequence, to set up experiments with a real vehicle, we must calculate the transformation between the six DOF controls and the controls corresponding to the thrusters. We address such a transformation for our actual test-bed vehicle in *(Chyba et al., 2008a,b)*.

Definition 4. The equations of motion for a general simple mechanical control system (rigid body) submerged in a real fluid subjected to external forces can be written as

$$\begin{aligned} \nabla_{\gamma'}\gamma' = &\, G^\# P(\gamma(t)) + \mathbb{G}^\#(F(\gamma'(t))) \\ &+ \sum_{i=1}^{6} \mathbb{I}_i^{-1}(\gamma(t))\sigma_i(t), \end{aligned} \quad (17)$$

where $G^\# P(\gamma(t))$ represents the potential force arising from gravity and the vehicle's buoyancy, $\mathbb{G}^\#(F(\gamma'(t)))$ represents the dissipative drag force, $\mathbb{I}_i^{-1} = \mathbb{G}^\# \pi_i = \mathbb{G}^{ij} X_j$, which may be represented as the i^{th} column of the matrix $\mathbb{I}^{-1} = \begin{pmatrix} M^{-1} & 0 \\ 0 & J^{-1} \end{pmatrix}$, and $\sigma_i(t)$ are the controls.

In the language of differential geometry, $G^\# P(\gamma(t))$ and $\mathbb{G}^\#(F(\gamma'(t)))$ are referred to as drift vector fields. These vector fields describe the dynamics of the system in the absence of controls.

The equations of motion given in Equation (17) are a coordinate invariant set of second-order, non-linear equations describing the dynamics of a submerged rigid body. These equations are equivalent to the following state equations as seen in *(Fossen, 1994)*, for example.

$$\dot{\boldsymbol{b}} = R\boldsymbol{\nu}, \quad (18)$$

$$\dot{R} = R\hat{\boldsymbol{\Omega}}, \quad (19)$$

$$M\dot{\boldsymbol{\nu}} = -\boldsymbol{\Omega} \times M\boldsymbol{\nu} + D_{\boldsymbol{\nu}}(\boldsymbol{\nu})\boldsymbol{\nu} - g(\boldsymbol{b}) + \boldsymbol{\varphi}_\nu, \quad (20)$$

$$J\dot{\boldsymbol{\Omega}} = -\boldsymbol{\Omega} \times J\boldsymbol{\Omega} - \boldsymbol{\nu} \times M\boldsymbol{\nu} + D_\Omega(\boldsymbol{\Omega})\boldsymbol{\Omega} - g(\boldsymbol{\eta}_2) + \boldsymbol{\tau}_\Omega. \quad (21)$$

Here, $\boldsymbol{\eta}_2 = (\phi, \theta, \psi)^t$, $g(\boldsymbol{b})$ and $g(\boldsymbol{\eta}_2)$ represent the restoring forces and moments and $\boldsymbol{\varphi}_\nu$ and $\boldsymbol{\tau}_\Omega$ account for the external control forces acting on the submerged rigid body.

[2] Note, C_D may not be dimensionless.

3.1 Ideal fluid

The above discussion details the general equations of motion for rigid body motion in a viscous fluid subject to external potential forces. For rigid body motion in an ideal fluid, the lack of viscosity allows us to neglect the external dissipative drag force. In the absence of this drag force, we can write the equations of motion as follows:

$$\nabla_{\gamma'}\gamma' = G^{\#}P(\gamma(t)) + \sum_{i=1}^{6} \mathbb{I}_i^{-1}(\gamma(t))\sigma_i(t). \tag{22}$$

Note here that we still include the potential forces. These equations, written as a first order system on TQ, take the form

$$\begin{aligned}\Upsilon'(t) &= S(\Upsilon(t)) + \mathrm{vlft}(G^{\#}P(\gamma(t)))(\Upsilon(t)) \\ &+ \sum_{i=1}^{m} \mathrm{vlft}\, \mathbb{I}_i^{-1}(\Upsilon(t))\sigma_i(t).\end{aligned} \tag{23}$$

Under the assumption that the rigid body is neutrally buoyant and that $C_B = C_G$, we can rewrite Equation (22) without the potential forces as

$$\nabla_{\gamma'}\gamma' = \sum_{i=1}^{6} \mathbb{I}_i^{-1}(\gamma(t))\sigma_i(t). \tag{24}$$

3.2 Real fluid

As mentioned earlier, the intent of this paper is to extend the notions of kinematic reduction and decoupling vector field to encompass potential and dissipative forces which occur in a real fluid. To this end, we begin by simplifying the general equations of motion given in Equation (17), but not to the extent of Equation (24). We assume that the rigid body is neutrally buoyant which implies that $W = B$. We also assume that $C_G = C_B$, which eliminates any righting moments. Under these assumptions, we have eliminated the potential forces acting on the vehicle and $G^{\#}P(\gamma(t)) = (0, 0, 0, 0, 0, 0)$. The only external force is the viscous drag term, and Equation (17) becomes

$$\nabla_{\gamma'}\gamma' = \mathbb{G}^{\#}(F(\gamma'(t))) + \sum_{i=1}^{6} \mathbb{I}_i^{-1}(\gamma(t))\sigma_i(t). \tag{25}$$

For the remainder of the paper, we will assume that Equation (25) represents the equations of motion for a submerged rigid body in a real fluid and Equation (24) corresponds to an ideal fluid. The inclusion of restoring forces is an extension of this paper and is considered in *(Smith, 2008)*.

3.3 Closing remark

In the sequel we denote by $\mathbb{I}^{-1}$ the matrix whose columns are the six input vector fields to the fully actuated system: $\mathbb{I}^{-1} = (\mathbb{I}_1^{-1}, \mathbb{I}_2^{-1}, \mathbb{I}_3^{-1}, \mathbb{I}_4^{-1}, \mathbb{I}_5^{-1}, \mathbb{I}_6^{-1})$. We also define $\mathcal{I}^{-1}_{n_1,\ldots n_m} = \{\mathbb{I}^{-1}_{n_1}, \ldots, \mathbb{I}^{-1}_{n_m}\} = \{\tilde{\mathbb{I}}_1^{-1}, \ldots, \tilde{\mathbb{I}}_m^{-1}\}$ to represent a set of input vector fields to an under-actuated system ($m < 6$). We note here that under our assumptions, $\mathbb{I}^{-1}$ is diagonal, and thus each $\mathbb{I}_i^{-1}$ and $\tilde{\mathbb{I}}_i^{-1}$, $i \in \{1, ..., 6\}$, represents a single degree of freedom input vector field to the system.

4. DECOUPLING VECTOR FIELDS FOR AN AUV IN AN IDEAL FLUID

In this paper, we are interested in finding solutions to the motion planning problem for the submerged rigid body. In the fully actuated case, as in Equation (24), this problem can be solved through the implementation of at most six pure motions. However, it is interesting to investigate the motion planning problem for an under-actuated vehicle. It has been shown in *(Bullo and Lynch, 2001)* that decoupling vector fields derived from a geometric reduction procedure may provide solutions to the motion planning problem in an under-actuated situation.

The affine connection control system defined in Equation (24) represents a second order control system on TQ, and we denote it by Σ_{dyn} to reiterate the fact that the control inputs are accelerations or dynamic inputs. The geometric reduction hinted at earlier is that of a kinematic reduction of Σ_{dyn} to a driftless system Σ_{kin} where the associated control system given by:

$$\gamma'(t) = \sum_{\alpha=1}^{m} \tilde{Z}_\alpha(\gamma(t))\sigma^\alpha_{kin}(t) \tag{26}$$

where $\{Z_1, ..., Z_\alpha\}$ are $\mathcal{C}^\infty$-setions of TQ, σ_{kin} are the associated kinematic controls and $m < 6$ denotes the rank of the reduction. If indeed, Σ_{kin} is a kinematic reduction of Σ_{dyn}, then for every controlled trajectory (γ, σ_{kin}) for Σ_{kin} there exists a dynamic control σ such that (γ, σ) is a controlled trajectory for Σ_{dyn}. A kinematic reduction of rank one is called a decoupling vector field.

In this section, we calculate the decoupling vector fields for each under-actuated scenario of a controlled submerged rigid body. By under-actuated we mean that the vehicle is unable to apply a control directly to one or more degrees of freedom. Practically, this is the situation of a distressed AUV which has lost power to one or more of its thrusters or actuators.

Since we consider the under-actuated scenario, we are not in the case where the input control vector fields are $\mathbb{I}^{-1} = \{\mathbb{I}_1^{-1}, ..., \mathbb{I}_6^{-1}\}$. Instead, the set of input control vector fields to the system are $\mathcal{I}^{-1}_{n_1,\ldots,n_m} = \{\tilde{\mathbb{I}}_1^{-1}, ..., \tilde{\mathbb{I}}_m^{-1}\}$ for $m < 6$, which is a subset of $\mathbb{I}^{-1}$. With these vector fields as inputs, and assuming that the body is neutrally buoyant with $C_G = C_B$, the equations of motion for the under-actuated system become

$$\nabla_{\gamma'}\gamma' = \sum_{i=1}^{m} \tilde{\mathbb{I}}_i^{-1}(\gamma(t))\sigma_i(t). \tag{27}$$

4.1 Covariant derivatives

To calculate decoupling vector fields, we must be able to calculate the covariant derivative of one vector field with respect to another. This section is devoted such computations. For the remainder of the paper, we assume that $\{X_1, ..., X_6\}$ is the standard basis for SE(3).

To calculate $\nabla_{\mathbb{I}_a^{-1}}\mathbb{I}_b^{-1}$, we use the following equation

$$
\begin{aligned}
&\mathbb{G}(\nabla_{\mathbb{I}_a^{-1}}\mathbb{I}_b^{-1}, X_k) = \\
&\frac{1}{2}[\mathcal{L}_{\mathbb{I}_a^{-1}}(\mathbb{G}(\mathbb{I}_b^{-1}, X_k)) + \mathcal{L}_{\mathbb{I}_b^{-1}}(\mathbb{G}(X_k, \mathbb{I}_a^{-1})) \\
&- \mathcal{L}_{X_k}(\mathbb{G}(\mathbb{I}_a^{-1}, \mathbb{I}_b^{-1})) + \mathbb{G}([\mathbb{I}_a^{-1}, \mathbb{I}_b^{-1}], X_k) \\
&- \mathbb{G}([\mathbb{I}_a^{-1}, X_k], \mathbb{I}_b^{-1}) - \mathbb{G}([\mathbb{I}_b^{-1}, X_k], \mathbb{I}_a^{-1})]
\end{aligned}
\tag{28}
$$

where $\mathbb{G}(X_i, X_j) = X_i^t \mathbb{G} X_j$ is the inner product which represents the kinetic energy metric on SE(3) . For any left-invariant frame field $Y_1, ..., Y_6$ on $SE(3)$, $\mathcal{L}_*(\mathbb{G}(*,*)) = 0$ so we need only calculate the last three terms in Equation (28). To calculate these terms, we need to know how to calculate the Lie bracket of vector fields. Since our configuration space is the Lie group SE(3), the linear space of body-fixed velocities is the Lie algebra $\mathfrak{se}(3)$:

$$
\mathfrak{se}(3) = \{\begin{bmatrix} 0 & 0 \\ \nu & \hat{\Omega} \end{bmatrix} | \nu \in \mathbb{R}^3,\ \Omega \in \mathbb{R}^3\}, \tag{29}
$$

where ν represents the translational velocities and Ω represents the rotational velocities. In the Lie algebra we know

$$
\begin{aligned}
&\left[\begin{bmatrix} 0 & 0 \\ \nu_1 & \hat{\Omega}_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \nu_2 & \hat{\Omega}_2 \end{bmatrix}\right] = \\
&\begin{bmatrix} 0 & 0 \\ \hat{\Omega}_1\nu_2 - \hat{\Omega}_2\nu_1 & \hat{\Omega}_1\hat{\Omega}_2 - \hat{\Omega}_2\hat{\Omega}_1 \end{bmatrix},
\end{aligned}
\tag{30}
$$

and since $\mathfrak{se}(3) \cong \mathbb{R}^3 \times \mathbb{R}^3 \ni (\nu, \Omega)$ we can write

$$
[(\nu_1, \Omega_1), (\nu_2, \Omega_2)] = (\Omega_1 \times \nu_2 - \Omega_2 \times \nu_1, \Omega_1 \times \Omega_2). \tag{31}
$$

Thus, we can define the adjoint operator $\mathrm{ad}_{(\nu,\Omega)} : \mathfrak{se}(3) \mapsto \mathfrak{se}(3)$ as $\mathrm{ad}_{(\nu_1,\Omega_1)}(\nu_2, \Omega_2) = [(\nu_1, \Omega_1), (\nu_2, \Omega_2)]$ and

$$
\mathrm{ad}_{(\nu,\Omega)} = \begin{bmatrix} \hat{\Omega} & \hat{\nu} \\ 0 & \hat{\Omega} \end{bmatrix}. \tag{32}
$$

By use of Equation (32), and remembering that our kinetic energy metric is

$$
\mathbb{G} = \mathrm{diag}(m_1, m_2, m_3, j_1, j_2, j_3) \tag{33}
$$

we have the equipment to calculate Equation (28) and calculate the covariant derivative between two input vector fields. The results of these calculations is presented in (Table 1). Since the results depend upon the symmetries of the vehicle, we will call our system kinetically unique if the inertial ellipsoid has no axes of symmetry. In particular, the added mass $(m + M_f^{\nu_i})$ and added moment of inertia $(J_{b_i} + J_f^{\Omega_i})$ coefficients are all distinct. We also introduce $\mathcal{U} = \{1, 2, 3\}$ and $\mathcal{V} = \{4, 5, 6\}$. We refer to $\mathbb{I}_i^{-1}, i \in \mathcal{U}$ as a translational control vector field and $\mathbb{I}_j^{-1}, j \in \mathcal{V}$ a rotational control vector field.

$(1,1)$	0	$(2,1)$	$\frac{-(m_1-m_2)X_6}{2j_3}$
$(1,2)$	$\frac{-(m_1-m_2)X_6}{2j_3}$	$(2,2)$	0
$(1,3)$	$\frac{-(m_3-m_1)X_5}{2j_2}$	$(2,3)$	$\frac{(m_3-m_2)X_4}{2j_1}$
$(1,4)$	0	$(2,4)$	$\frac{-(m_3-m_2)X_3}{2m_3}$
$(1,5)$	$\frac{(m_3-m_1)X_3}{2m_3}$	$(2,5)$	0
$(1,6)$	$\frac{(m_1-m_2)X_2}{2m_2}$	$(2,6)$	$\frac{(m_1-m_2)X_1}{2m_1}$
$(3,1)$	$\frac{-(m_3-m_1)X_5}{2j_2}$	$(4,1)$	0
$(3,2)$	$\frac{(m_3-m_2)X_4}{2j_1}$	$(4,2)$	$\frac{(m_3+m_2)X_3}{2m_3}$
$(3,3)$	0	$(4,3)$	$\frac{-(m_3+m_2)X_2}{2m_2}$
$(3,4)$	$\frac{-(m_3-m_2)X_2}{2m_2}$	$(4,4)$	0
$(3,5)$	$\frac{(m_3-m_1)X_1}{2m_1}$	$(4,5)$	$\frac{(j_3+j_2-j_1)X_6}{2j_3}$
$(3,6)$	0	$(4,6)$	$\frac{-(j_3+j_2-j_1)X_5}{2j_2}$
$(5,1)$	$\frac{-(m_3+m_1)X_3}{2m_3}$	$(6,1)$	$\frac{(m_2+m_1)X_2}{2m_2}$
$(5,2)$	0	$(6,2)$	$\frac{-(m_2+m_1)X_1}{2m_1}$
$(5,3)$	$\frac{(m_3+m_1)X_1}{2m_1}$	$(6,3)$	0
$(5,4)$	$\frac{-(j_3-j_2+j_1)X_6}{2j_3}$	$(6,4)$	$\frac{-(j_3-j_2-j_1)X_5}{2j_2}$
$(5,5)$	0	$(6,5)$	$\frac{(j_3-j_2-j_1)X_4}{2j_1}$
$(5,6)$	$\frac{(j_3-j_2+j_1)X_4}{2j_1}$	$(6,6)$	0

Table 1. Covariant derivatives in basis notation for the Levi-Civita connection ∇. We use $(i,j) = \nabla_{\mathbb{I}_i^{-1}}\mathbb{I}_j^{-1}$.

4.2 Computing decoupling vector fields in an ideal fluid

By use of (Table 1), we can calculate the decoupling vector fields for every under-acutated scenario for a rigid-body submerged in an ideal fluid. After introducing some additional terminology, we display these results in a Proposition.

Definition 5. A vector field V is called an *axial field* if it is of the form $V = h_i\,\mathbb{I}_i^{-1} + h_{i+3}\,\mathbb{I}_{i+3}^{-1}$ where $i \in \mathcal{U}$.

We use the term axial field since the integral curves are a translation and rotation acting on the same principal axis of inertia. We call the integral curves of such a vector field *axial motions*.

Definition 6. A vector field V is called a *coordinate field* if it is of the form $V = h_i\,\mathbb{I}_i^{-1} + h_j\,\mathbb{I}_j^{-1} + h_k\,\mathbb{I}_k^{-1}$ where $i = 1$ or 4, $j = 2$ or 5 and $k = 3$ or 6.

We choose the term coordinate field since all three principal axes of the inertial coordinate frame are represented. An integral curve for such a vector field is referred to as a *coordinate motion*. With this information, we can state the following Proposition.

Proposition 7. Under our assumptions on a submerged rigid body in an ideal fluid we have the following characterization for the decoupling vector fields in terms of the number of degrees of freedom we can input to the system.

Case 1: *Single-input system*, $\mathcal{I}_{n_1}^{-1} = \{\tilde{\mathbb{I}}_1^{-1}\}$. The decoupling vector fields are multiples of $\tilde{\mathbb{I}}_1^{-1}$; these are pure motions.

Case 2: *Two-input system*, $\mathcal{I}_{n_1,n_2}^{-1} = \{\tilde{\mathbb{I}}_1^{-1}, \tilde{\mathbb{I}}_2^{-1}\}$ in which both inputs do not act upon the same principle axis of inertia. Then, for a kinetically unique system, a vector field $V \in \mathcal{I}^{-1}$ is decoupling if and only if V has all but one of its components equal to zero. In particular, it has the form $V = h_1\,\tilde{\mathbb{I}}_1^{-1}$ or $V = h_2\,\tilde{\mathbb{I}}_2^{-1}$; these are pure motions. If the input vector fields act on the same principal axis of inertia, then every vector field in $\mathcal{I}$ is decoupling.

Case 3: *Three-input system.*

(1) *Three Translational Inputs*: $\mathcal{I}_{1,2,3}^{-1} = \{\mathbb{I}_1^{-1}, \mathbb{I}_2^{-1}, \mathbb{I}_3^{-1}\}$. For a kinetically unique system, a vector field $V \in \mathcal{I}^{-1}$ is decoupling if and only if V has all but one of its components equal to zero. In particular, it has the form $V = h_i\,\mathbb{I}_i^{-1}$ for $i \in \mathcal{U}$; these are the pure translational motions. Assuming exactly two of the m_i's are equal, we get the axial motions as additional decoupling vector fields: $V = h_i\,\mathbb{I}_i^{-1} + h_j\,\mathbb{I}_j^{-1}$, where $m_i = m_j$ and $m_i \neq m_k$. If $m_i = m_j =$

m_k, then every vector field $V \in \mathcal{I}^{-1}$ is decoupling since $\nabla_V V \in \mathcal{I}^{-1}$.

(2) *Three Rotational Inputs*: $\mathcal{I}^{-1}_{4,5,6} = \{\mathbb{I}^{-1}_4, \mathbb{I}^{-1}_5, \mathbb{I}^{-1}_6\}$. In this case $\nabla_V V \in \mathcal{I}^{-1}$ for all $V \in \mathcal{I}^{-1}$, thus each vector field $V \in \mathcal{I}^{-1}$ is decoupling.

(3) *Mixed Translational and Rotational Inputs.* Suppose we have a kinetically unique three input system such that the inputs are not all translational or all rotational but represents motions along three distinct axis. Then, every vector field $V \in \mathcal{I}^{-1}$ is decoupling. If the three input system is such that it represents an axial motion plus another input vector field, the decoupling vector fields are the axial motions, $V = h_i\,\mathbb{I}^{-1}_i + h_{i+3}\,\mathbb{I}^{-1}_{i+3}$ for $i \in \mathcal{U}$, and the pure motions, $V = h_j\,\mathbb{I}^{-1}_j$ where $j \neq i$ and $j \neq i+3$. The remarks about the symmetries in the case of three translational input are valid in this case also.

Case 4: *Four input system.*

(1) *Three Translation, One Rotation*: $\mathcal{I}^{-1}_{1,2,3,k} = \{\mathbb{I}^{-1}_1, \mathbb{I}^{-1}_2, \mathbb{I}^{-1}_3, \mathbb{I}^{-1}_k\}$ where $k \in \mathcal{V}$. For a kinetically unique system the decoupling vector fields are the axial motions $V = h_{k-3}\,\mathbb{I}^{-1}_{k-3} + h_k\,\mathbb{I}^{-1}_k$ or the coordinate motions $V = h_i\,\mathbb{I}^{-1}_i + h_j\,\mathbb{I}^{-1}_j + h_k\,\mathbb{I}^{-1}_k$ with $i,j \in \mathcal{U}$, $i,j \neq k-3$. If $m_{k-3} = m_i$ for $i \in \mathcal{U}$ and $i \neq k-3$, then $V = h_i\,\mathbb{I}^{-1}_i + h_{k-3}\,\mathbb{I}^{-1}_{k-3} + h_k\,\mathbb{I}^{-1}_k$ is also a decoupling vector field. If $m_1 = m_2 = m_3$, then every vector field $V \in \mathcal{I}^{-1}$ is a decoupling vector field.

(2) *Three Rotations, One Translation*: $\mathcal{I}^{-1}_{i,4,5,6} = \{\mathbb{I}^{-1}_i, \mathbb{I}^{-1}_4, \mathbb{I}^{-1}_5, \mathbb{I}^{-1}_6\}$ where $i \in \mathcal{U}$. Then the decoupling vector fields are the axial motions $V = h_i\,\mathbb{I}^{-1}_i + h_{i+3}\,\mathbb{I}^{-1}_{i+3}$ or the coordinate motions $V = h_4\,\mathbb{I}^{-1}_4 + h_5\,\mathbb{I}^{-1}_5 + h_6\,\mathbb{I}^{-1}_6$.

(3) *Two Translations, Two Rotations.* For a kinetically unique system, if two principle axes are repeated: $\mathcal{I}^{-1}_{i,j,i+3,j+3} = \{\mathbb{I}^{-1}_i, \mathbb{I}^{-1}_j, \mathbb{I}^{-1}_{i+3}, \mathbb{I}^{-1}_{j+3}\}$ where $i,j \in \mathcal{U}$, then the decoupling vector fields are either the pure motions $V = h_a\,\mathbb{I}^{-1}_a$ for $a \in \{i,j,i+3,j+3\}$ or the axial motions $V = h_a\,\mathbb{I}^{-1}_a + h_{a+3}\,\mathbb{I}^{-1}_{a+3}$ where $a = 1$ or $a = j$. If $m_i = m_j$, then additional decoupling vector fields for the system are the axial motions plus a multiple of the other translational input vector field: $V = h_i\,\mathbb{I}^{-1}_i + h_j\,\mathbb{I}^{-1}_j + h_k\,\mathbb{I}^{-1}_k$ where $k = i+3$ or $k = j+3$. And, if $j_i = j_j$, then additional decoupling vector fields for the system are of the form $V = h_{i+3}\,\mathbb{I}^{-1}_{i+3} + h_{j+3}\,\mathbb{I}^{-1}_{j+3}$. For a kinetically unique system, if one principle axis is repeated: $\mathcal{I}^{-1}_{i,j,i+3,k+3} = \{\mathbb{I}^{-1}_i, \mathbb{I}^{-1}_j, \mathbb{I}^{-1}_{i+3}, \mathbb{I}^{-1}_{k+3}\}$ where $i,j,k \in \mathcal{U}$, then the decoupling vector fields are the axial motions $V = h_i\,\mathbb{I}^{-1}_i + h_{i+3}\,\mathbb{I}^{-1}_{i+3}$ or the coordinate motions $V = h_i\,\mathbb{I}^{-1}_i + h_j\,\mathbb{I}^{-1}_j + h_{k+3}\,\mathbb{I}^{-1}_{k+3}$. If $j_i = j_k$ then h_j or h_{i+3} must be zero, and additional decoupling vector fields are the axial motions plus a multiple of the other rotational input vector field: $V = h_i\,\mathbb{I}^{-1}_i + h_{i+3}\,\mathbb{I}^{-1}_{i+3} + h_{k+3}\,\mathbb{I}^{-1}_{k+3}$.

Case 5: *Five input system.*

(1) *Three Translations, Two Rotations*: $\mathcal{I}^{-1}_{1,2,3,i,j} = \{\mathbb{I}^{-1}_1, \mathbb{I}^{-1}_2, \mathbb{I}^{-1}_3, \mathbb{I}^{-1}_i, \mathbb{I}^{-1}_j\}$ where $i,j \in \mathcal{V}$, and let $k \in \mathcal{V}$ such that $k \neq i$ *or* j. For a kinetically unique system the decoupling vector fields are:

(a) The axial motions plus a multiple of a translational input $V = h_a\,\mathbb{I}^{-1}_a + h_{a+3}\,\mathbb{I}^{-1}_{a+3} + h_{k-3}\,\mathbb{I}^{-1}_{k-3}$ where $a \in \mathcal{U} - (k-3)$.

(b) The coordinate motions $V = h_a\,\mathbb{I}^{-1}_a + h_b\,\mathbb{I}^{-1}_b + h_{k-3}\,\mathbb{I}^{-1}_{k-3}$ where $a,b \in \mathcal{U} - (k-3)$.

(c) The motions defined by $V = h_{k-3}\,\mathbb{I}^{-1}_{k-3} + h_b\,\mathbb{I}^{-1}_b$ where $b \in \mathcal{U} - (k-3)$.

(d) The pure motion $V = h_{k-3}\,\mathbb{I}^{-1}_{k-3}$

Assuming that $m_{i-3} = m_{j-3}$, additional decoupling vector fields are given by $V = h_a\,\mathbb{I}^{-1}_a + h_k\,\mathbb{I}^{-1}_k + h_i\,\mathbb{I}^{-1}_i + h_j\,\mathbb{I}^{-1}_j$ where $a = i-3$ or $a = j-3$ and $k \in \mathcal{U} - \{i-3, j-3\}$. Assuming that $j_{i-3} = j_{j-3}$, additional decoupling vector fields are given by $V = h_1\,\mathbb{I}^{-1}_1 + h_2\,\mathbb{I}^{-1}_2 + h_3\,\mathbb{I}^{-1}_3 + h_a\,\mathbb{I}^{-1}_a$ where $a = i$ or $a = j$.

(2) *Two Translations, Three Rotations*: $\mathcal{I}^{-1}_{i,j,4,5,6} = \{\mathbb{I}^{-1}_i, \mathbb{I}^{-1}_j, \mathbb{I}^{-1}_4, \mathbb{I}^{-1}_5, \mathbb{I}^{-1}_6\}$ where $i,j \in \mathcal{U}$, and let $k \in \mathcal{U}$ such that $k \neq i$ *or* j. Regardless whether this system is kinetically unique or not, the decoupling vector fields are:

(a) The axial motions plus a multiple of a rotational input $V = h_a\,\mathbb{I}^{-1}_a + h_{a+3}\,\mathbb{I}^{-1}_{a+3} + h_{k+3}\,\mathbb{I}^{-1}_{k+3}$ where $a \in \mathcal{V} - (k+3)$.

(b) The coordinate motions $V = h_a\,\mathbb{I}^{-1}_a + h_b\,\mathbb{I}^{-1}_b + h_{k+3}\,\mathbb{I}^{-1}_{k+3}$ where $a,b \in \mathcal{V} - (k+3)$.

(c) The motions defined by $V = h_{k+3}\,\mathbb{I}^{-1}_{k+3} + h_b\,\mathbb{I}^{-1}_b$ where $b \in \mathcal{V} - (k+3)$.

(d) The pure motion $V = h_{k+3}\,\mathbb{I}^{-1}_{k+3}$.

Case 6: *Six input system.* Every vector field is decoupling.

Proof: The results are directly computational and can be found in Chapter 5 and Appendix *C* of *(Smith, 2008)*.

5. DECOUPLING VECTOR FIELDS FOR AN AUV IN A REAL FLUID

In Section 3, we presented the respective equations of motion for a rigid body submerged in either a real fluid and an ideal fluid. The presentations use the Levi-Civita affine connection. In Section 4 we calculate the decoupling vector fields for a test-bed AUV submerged in an ideal fluid without external forces. For practical application concerns, we wish to apply the theory of kinematic reductions and decoupling vector fields, as presented earlier, to the forced affine connection control system. However, given the form of Equation (25), we are unable to directly apply the results of the previous sections; the external forces add a drift vector field to the geometric acceleration and thus we cannot use the defined kinematic reduction.

To eliminate the restoring forces, at least temporarily, we will still assume that the vessel is neutrally buoyant and that $C_G = C_B$, but we now consider a viscous fluid. As previously assumed, based on the vehicle and the motions considered, the magnitude of the viscous drag force and moments acting on a rigid body are proportional to the square of the velocity of the body. This relationship allows us to describe the geometric acceleration associated to the viscous drag forces and moments as a symmetric type $(1,2)$-tensor field on $\mathbb{R}^3 \times \mathrm{SO}(3)$.

$(1,1)$	$-\frac{D_1}{m_1}X_1$	$(2,2)$	$-\frac{D_2}{m_2}X_2$
$(3,3)$	$-\frac{D_3}{m_3}X_3$	$(4,4)$	$-\frac{D_4}{j_1}X_4$
$(5,5)$	$-\frac{D_5}{j_2}X_5$	$(6,6)$	$-\frac{D_6}{j_3}X_6$

Table 2. Covariant derivatives in basis notation for the connection $\widetilde{\nabla}$ for the case when $i = j$.

Since the difference between any two affine connections is a type $(1,2)$-tensor field Δ, we can incorporate viscous drag into a new connection $\widetilde{\nabla}$ by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \Delta(X,Y). \tag{34}$$

In general, a symmetric type (1,2)-tensor is given by

$$\Delta = \sum_{i,j,k} \Delta^i_{jk} X_i \otimes \pi^j \otimes \pi^k, \tag{35}$$

where $\Delta^i_k j = \Delta^i_j k$ and thus when we evaluate Δ along the trajectory we get

$$\begin{aligned}\Delta|_{(b,R)}(\gamma', \gamma') = \sum_{i=1}^{6}\Big(&\sum_{j,k=1}^{3} \Delta^i_{jk}(b,R)\nu^j\nu^k \\ &+ 2\sum_{j=1}^{3}\sum_{k=4}^{6}\Delta^i_{jk}(b,R)\nu^j\Omega^k \\ &+ \sum_{j,k=4}^{6}\Delta^i_{jk}(b,R)\Omega^j\Omega^k\Big)X_i.\end{aligned} \tag{36}$$

Computing the above using the basis $\{X_1, ..., X_6\}$ from before, we get that

$$\Delta(X_i, X_j) = \Delta^1_{ij}X_1 + ... + \Delta^6_{ij}X_6. \tag{37}$$

Using Equation (16), we have that $\Delta(X_i, X_i) = \Delta^i_{ii}X_i = -\frac{D_i}{\mathbb{G}_{ii}}X_i$. Thus, we are able to define the new connection in the following way

$$\widetilde{\nabla}_{X_i}X_j = \begin{cases} -\dfrac{D_i}{\mathbb{G}_{ii}}X_i, & i = j, \\ \nabla_{X_i}X_j, & i \neq j, \end{cases} \tag{38}$$

since we know $\nabla_{X_i}X_i = 0$. With this new connection, the equations of motion for the forced affine connection control system become

$$\widetilde{\nabla}_{\gamma'}\gamma' = \sum_{a=1}^{6}\sigma^a(t)\mathbb{I}_a^{-1}(\gamma(t)). \tag{39}$$

The above system is a second order affine connection control system on Q, just as we saw in Section 3. At first it looks as though we neglect the drag forces, but they are now hidden in the new connection $\widetilde{\nabla}$. Note that we have essentially altered the acceleration of the system to account for the dissipation due to viscous drag.

By use of the new affine connection $\widetilde{\nabla}$, we can calculate the covariant derivatives of the input vector fields in basis notation corresponding to motion in a real fluid. By use of Equation (38), the results will be the same as for the Levi-Civita affine connection except for the case of $\widetilde{\nabla}_{\mathbb{I}_i^{-1}}\mathbb{I}_i^{-1}$. These six covariant derivatives are presented in (Table 2). Here the drag term $\frac{D_i}{\mathbb{G}_{ii}}$ shows up, but has no impact on the computation of decoupling vector fields in the real fluid scenario.

Proposition 8. By use of the Levi-Civita affine connection, we can define an affine connection $\widetilde{\nabla}$ which includes external dissipative drag terms for the submerged rigid body and has the property that:

$$\widetilde{\nabla}_{\mathbb{I}_i^{-1}}\mathbb{I}_j^{-1} = \begin{cases} \nabla_{\mathbb{I}_i^{-1}}\mathbb{I}_j^{-1}, & i \neq j, \\ -\dfrac{D_i}{\mathbb{G}_{ii}}X_i, & i = j. \end{cases}$$

Proof: Apply the results of Equation (38) to the control input vector fields $\mathbb{I}_k^{-1}$ for $k = 1, ..., 6$.

Theorem 9. The decoupling vector fields for a rigid body submerged in a real fluid calculated using the connection $\widetilde{\nabla}$ are the same as for a rigid body submerged in an ideal fluid.

Proof: Using Proposition 8 and the fact that $\nabla_{\mathbb{I}_i^{-1}}\mathbb{I}_i^{-1} = -\frac{D_i}{\mathbb{G}_{ii}}X_i \in \operatorname{Span}\mathcal{I}^{-1}_{n_1,\ldots,n_k}$ gives the result.

6. MOTION PLANNING USING KINEMATIC MOTIONS

In this section, we present some practical applications to demonstrate the use of decoupling vector fields in the motion planning problem for AUVs. For these examples, we will consider a typical torpedo-shaped AUV, similar to that of Hydroid, LLC's Remus AUVs[3] or one of MIT's Odyssey II series *(MIT, 2007)*. We assume that the AUV is submerged in a viscous fluid and subject to dissipative drag forces. We will use the $\tilde{}$ notation where appropriate to denote the use of or association with the modified connection $\tilde{\nabla}$ defined in Section 5. For examples of kinematic motion planning for another type of AUV, see *(Chyba et al., 2008c)* or *(Smith, 2008)*. We consider a few control schemes for the torpedo-shaped AUV and calculate their corresponding decoupling vector fields. We also consider some applications which are best suited for this type of motion planning.

We are now ready to discuss the procedure of motion planning via kinematic motions. First, we begin with initial (η_{init}) and final (η_f) configurations for the system. At this point, we must determine whether or not the final configuration is reachable using only the kinematic motions defined by the decoupling vector fields. This can be done by showing that the system is kinematically controllable, or by producing a concatenation of kinematic motions connecting η_{init} and η_f. In either case, we solve the motion planning problem for Σ_{kin} by concatenating the flows of the given decoupling vector fields from η_{init} and η_f. For each concatenated section of the kinematic motion, we reparameterize the integral curve such that the initial and final velocities are zero. This then gives us the kinematic controls σ_{kin} with which to realize the desired motion. We then calculate the dynamic controls using Theorem 13.5 from *(Bullo and Lewis, 2005a)*. Similar applications of this type of motion planning related to AUVs can be found in *(Chyba and Smith, 2008)* and *(Smith, 2008)*. We conclude this section with practical applications.

---
[3] Information regarding REMUS vehicles can be found at http://www.hydroidinc.com/

For the following calculations, we consider a torpedo-shaped AUV similar to a standard REMUS vehicle [4]. We simplify the model a bit by assuming that $C_G = C_B$, the vehicle is neutrally buoyant and has three planes of symmetry.

For the first scenario, we assume that we can control surge (propeller in the rear), pitch (foil in front or rear) and yaw (rudder in rear). Using a rudder and and a foil to control pitch and yaw implies that we must also include a surge component to realize these rotations. This is different from the discussion in Remark 3 since the forces do not all come from physical thrusters, however, we may still assume that these forces act at C_G. Understanding this view of the control forces, we take the input vector fields for the mechanical system to be $\mathbb{T}_1 = (\frac{1}{m_1}, 0, 0, 0, 0, 0)^t$, $\mathbb{T}_2 = (\frac{1}{m_1}, 0, 0, 0, \frac{\phi}{j_2}, 0)^t$, and $\mathbb{T}_3 = (\frac{1}{m_1}, 0, 0, 0, 0, \frac{\psi}{j_3})^t$. Here the ϕ and ψ represent the angle of rotation that the vehicle experiences after one time step. In this simplified scenario, in order to realize a rotation, we fix the fins or rudders in a given position and apply a surge which implements the rotation along with the surge motion.

Recall from the definition of an affine connection, that the map $(X, Y) \mapsto \widetilde{\nabla}_X Y$ is $\mathbb{R}$-bilinear. Thus, $\widetilde{\nabla}_{W+X}(Y + Z) = \widetilde{\nabla}_W(Y+Z) + \widetilde{\nabla}_X(Y+Z) = \widetilde{\nabla}_W Y + \widetilde{\nabla}_W Z + \widetilde{\nabla}_X Y + \widetilde{\nabla}_X Z$ for vector fields W, X, Y, Z. This bilinearity suggests that we should express the input vector fields as $\mathbb{T}_1 = \mathbb{I}_1^{-1}$, $\mathbb{T}_2 = \mathbb{I}_1^{-1} + \phi\mathbb{I}_5^{-1}$ and $\mathbb{T}_3 = \mathbb{I}_1^{-1} + \psi\mathbb{I}_6^{-1}$, where $\{\mathbb{I}_1^{-1}, ..., \mathbb{I}_6^{-1}\}$ are input vector fields as defined in Definition 4.

With the input vector fields rewritten, we can now calculate the decoupling vector fields for the system as previously shown. Suppose that $V \in \text{Span}\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3\}$ is a decoupling vector field for the system. Using the tools from Sections 4.2 and 5, a direct computation for this three input system shows that,

$$\begin{aligned}\widetilde{\nabla}_V V = \frac{1}{2}(&h_1^2(2\widetilde{\nabla}_{\mathbb{T}_1}\mathbb{T}_1) + h_2^2(2\widetilde{\nabla}_{\mathbb{T}_2}\mathbb{T}_2) + h_3^2(2\widetilde{\nabla}_{\mathbb{T}_3}\mathbb{T}_3) \\ &+ h_1h_2(\widetilde{\nabla}_{\mathbb{T}_1}\mathbb{T}_2 + \widetilde{\nabla}_{\mathbb{T}_2}\mathbb{T}_1) + h_1h_3(\widetilde{\nabla}_{\mathbb{T}_1}\mathbb{T}_3 + \widetilde{\nabla}_{\mathbb{T}_3}\mathbb{T}_1) \\ &+ h_2h_3(\widetilde{\nabla}_{\mathbb{T}_2}\mathbb{T}_3 + \widetilde{\nabla}_{\mathbb{T}_3}\mathbb{T}_2)).\end{aligned} \tag{40}$$

Remember that, in order to be decoupling, a vector field V as well as $\widetilde{\nabla}_V V$ must be in the span of the input vector fields. If we consider just the first line of Equation (40), we get

$$\begin{aligned}&\frac{1}{2}(2h_1^2(-\frac{D_1}{m_1}\mathbb{I}_1^{-1}) + 2h_2^2(-\frac{D_1}{m_1}\mathbb{I}_1^{-1} - \frac{m_1}{m_3}\mathbb{I}_3^{-1} - \frac{D_5}{j_2}\mathbb{I}_5^{-1}) \\ &+ 2h_3^2(-\frac{D_1}{m_1}\mathbb{I}_1^{-1} + \frac{m_1}{m_2}\mathbb{I}_2^{-1} - \frac{D_6}{j_3}\mathbb{I}_6^{-1}),\end{aligned} \tag{41}$$

which clearly shows that both h_2 and h_3 must be zero since $\mathbb{I}_3^{-1}$ and $\mathbb{I}_2^{-1}$ are not in $\text{Span}\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3\}$. Further computation shows that $\mathbb{T}_1$ is a decoupling vector field, and is the only decoupling vector field for this system. The only kinematic motion for the system is a pure surge. Thus, this scenario does not lend well to motion planning using kinematic motions.

[4] Information regarding REMUS vehicles can be found at http://www.hydroidinc.com/

As a second scenario, we consider an operational situation in which the vehicle has a rear propeller to provide surge and independently actuated fins to control roll, pitch and yaw. Thus, we take the input control vector fields to be $\mathbb{T}_1 = (\frac{1}{m_1}, 0, 0, 0, 0, 0)^t$, $\mathbb{T}_2 = (\frac{1}{m_1}, 0, 0, 0, \frac{\phi}{j_2}, 0)^t$, $\mathbb{T}_3 = (\frac{1}{m_1}, 0, 0, 0, 0, \frac{\psi}{j_3})^t$ and $\mathbb{T}_4 = (\frac{1}{m_1}, 0, 0, -\frac{\theta}{j_1}, 0, 0)^t$. This case is analyzed similarly to the previous case. The calculations yield the axial vector field $V = h_1\mathbb{T}_1 + h_4\mathbb{T}_4$ as decoupling. This scenario produces the kinematic motions defined by a pure surge, a pure roll or a simultaneous roll and surge motion.

The first two examples served to demonstrate the procedure for calculating decoupling vector fields for a given under-actuated system. In this next example, we focus on finding a realizable trajectory and calculating the control strategy required to realize the motion. To calculate the control strategy for this mission, we use published values for the hydrodynamic parameters of a typical REMUS-type vehicle. The kinetic energy metric is assumed to be diagonal and is given by

$$\begin{aligned}\mathbb{G} &= \text{diag}(m_1, m_2, m_3, j_1, j_2, j_3) \\ &= \text{diag}(31.43, 66, 66, 0.0704, 4.88, 4.88),\end{aligned} \tag{42}$$

where the first terms are the translational added mass terms with units of kg and the final three are the rotational added mass terms with units $\frac{kg \cdot m^2}{rad^2}$. The six principal drag coefficients are given by $\{D_1 = 3.9\frac{kg}{m}, D_2 = 131\frac{kg}{m}, D_3 = 131\frac{kg}{m}, D_4 = 0.13\frac{kg \cdot m^2}{rad^2}, D_5 = 188\frac{kg \cdot m^2}{rad^2}, D_6 = 94\frac{kg \cdot m^2}{rad^2}\}$. The mass of the vehicle is assumed to be $30.5kg$. A typical velocity for the vehicle is 1.5m/s (3 knots). Hydrodynamic parameters and coefficients for the considered vehicle were taken from *(Prestero, 2001)*.

For this mission, we will assume that the vehicle is neutrally buoyant, $C_G = C_B$ and we have a rear propeller to provide surge and an actuated rudder to control the yaw motion. The input control vector fields are then given by $\mathbb{T}_1 = (\frac{1}{m_1}, 0, 0, 0, 0, 0)^t$ and $\mathbb{T}_5 = (\frac{\cos\alpha}{m_1}, \frac{\sin\alpha}{m_2}, 0, 0, 0, \frac{-d\sin\alpha}{j_3})^t$, where α is the fixed angle of the rudder, $d = 0.7$m is the moment arm measured from C_G.

Performing a similar analysis to that done in the previous examples, we find that the decoupling vector fields for the system are $\mathbb{T}_1, \mathbb{T}_5$ and the linear combinations of these two vector fields. Hence, given the control inputs to the system, we can only follow the integral curves of $\mathbb{T}_1, \mathbb{T}_5$ or their linear combinations to design a trajectory; effectively we can travel in a straight line or an elliptical path based on the angle of the rudder. As a practical application, let us consider the mission of searching for a lost vessel. In this scenario, we would have a general area, say 10km×10km, within which to search. One solution is to use the exhaustive back and forth search to photograph the entire area using side-scan sonar or CCD cameras for later review. This mission requires the vehicle to realize a sequence of parallel paths connected at the ends by a turn around loop; just as you would imagine mowing a rectangular lawn. The calculated decoupling vector fields allow us to perform such a mission using the trajectory displayed in Figure 2. Concatenating many of these transects together will define a search mission for the vehicle. The next step is to design the control strategy for this mission.

As previously mentioned, the controls are calculated by use of Theorem 13.5 in *(Bullo and Lewis, 2005a)*, and each concatenated section is parameterized to begin and end with zero velocity. The general idea is to follow the integral curves of $\mathbb{T}_1$ to realize the main portion of the transect, and then follow the integral curves of $\mathbb{T}_5$ to complete the turn around at the end, then follow the integral curves of $\mathbb{T}_1$ again to complete another transect. Since there are only two input control vector fields for this system, we need only calculate two controls, $u^1(t)$ and $u^2(t)$ corresponding to $\mathbb{T}_1$ and $\mathbb{T}_5$, respectively.

For the first leg of this mission, we choose the decoupling vector field to be $V_1 = (1,0,0,0,0,0)^t$. This will allow us to complete the transect in 10000 seconds. Based on a vehicle velocity of 1.6m/s, we reparameterize the motion to begin and end with zero velocity and have a duration of 6250 seconds. For the turn around section of the trajectory, we take $V_2 = (0.2, 0.1, 0, 0, 0, -1)^t = 10\mathbb{T}_5$ as the decoupling vector field. We parameterize this motion to last for 30 seconds. The final transect leg is completed by applying the same control as computed for the first leg of the mission. The entire mission has a total duration of 12,530 seconds. The control strategies are presented in Equations (43) and (44). The $u^1(t)$ control is plotted in (Figure 3) and (Figure 4). The $u^2(t)$ control is plotted in (Figure 5). For these figures, the time is displayed in seconds and the controls $u^i(t)$ have units of Newtons. We remark that for this example we do not explicitly account for thruster saturation, however we do utilize the operational velocity of the vehicle for our reparameterization which implicitly accounts for any bounds on the actuators.

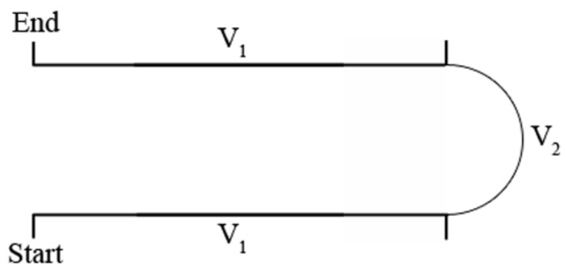

Fig. 2. *Two transects connected by turn around loop.*

$$u^1(t) = \begin{cases} \frac{6}{11920928955}(0.0005t^4 \\ -0.59t^3 + 18281t^2 \\ -30693t + 95916748), & t \in [0, 6250), \\ \\ -1.5 \times 10^{-16}(4.1 \times 10^{11}t^4 \\ -1 \times 10^{16}t^3 + 9.7 \times 10^{18}t^2 \\ -4 \times 10^{23}t + 6.3 \times 10^{26}), & t \in [6250, 6280), \\ \\ 5 \times 10^{-16}(468t^4 - 1.8 \times 10^7 t^3 \\ +2.4 \times 10^{11}t^2 - 1.4 \times 10^{15}t \\ +2.9 \times 10^{18}), & t \in [6280, 12530). \end{cases} \quad (43)$$

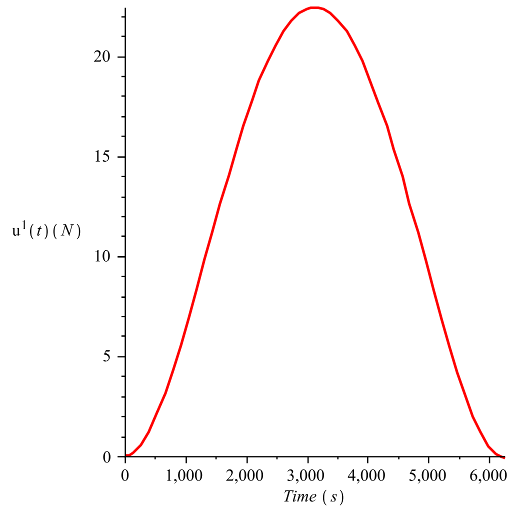

Fig. 3. Continuous control strategy for the input vector field $V_1 = \mathbb{T}_1$ to realize the straight-line transect.

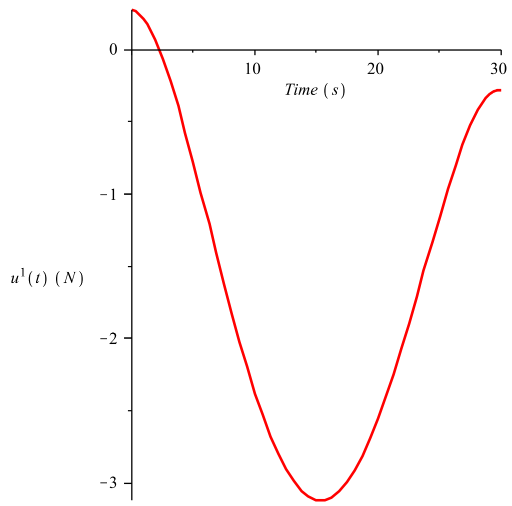

Fig. 4. Continuous control strategy for u^1 for the input vector field $V_2 = 10\mathbb{T}_5$ to realize the turn-around.

$$u^2(t) = \begin{cases} 0, & t \in [0, 6250), \\ \\ 1.85 \times 10^{-17}(4.96 \times 10^{12}t^4 \\ -1 \times 10^{16}t^3 + 9.7 \times 10^{18}t^2 \\ -4 \times 10^{23}t + 6.3 \times 10^{26}), & t \in [6250, 6280), \\ \\ 0, & t \in [6280, 12530]. \end{cases} \quad (44)$$

7. CONCLUSION

The goal of this research is to provide solutions to the motion planning problem for a rigid body submerged in a real fluid. The presented technique also provides motion planning solutions for vehicles operating in an under-actuated condition. Both of these scenarios are useful in motion planning for practical applications.

This paper makes the first step towards merging the theory and application by including dissipative viscous

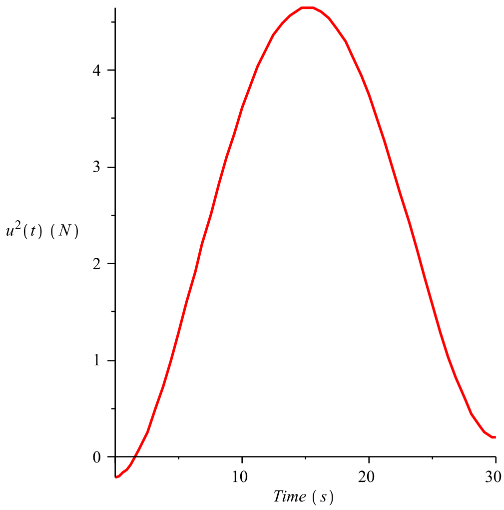

Fig. 5. Continuous control strategy for u^2 for the input vector field $V_2 = 10\mathbb{T}_5$ to realize the turn-around.

drag into the geometric formulation. The next step for the extension will be to include forces and moments arising from a potential function such as buoyancy and righting moments as well as environmental disturbances such as ocean currents. Since this research focuses on both the theory and application, a well built extension from the ideal fluid case to the real fluid case would be quite beneficial.

We note that for a real AUV, the assumption of a neutrally buoyant vehicle is not impractical. However, the assumption of $C_G = C_B$ is generally not the case in practice, since their separation provides stability for the vehicle. This separation of C_G and C_B creates righting forces and moments when a vehicle lists. These restorative forces are currently under investigation to provide a more realistic extension of the geometric control theory presented in this paper. Preliminary work in this area is presented in *(Smith et al., 2008)* and *(Smith, 2008)*.

Even when neglecting restoring forces, it is obvious that motion planning via kinematic motions is a useful tool for AUV applications. The decoupling vector fields provide insight into the physical design of an AUV. We can calculate the minimal number of input control vector fields for a system, such that the vehicle is still able to reach any given configuration. This informs us how to efficiently construct AUVs with respect to the physical actuators which control its motion. This same analysis of decoupling vector fields can plan the motion for a vehicle to get back home in a damaged situation. Examining both of these scenarios allows the design team to implement only necessary redundancy which saves time and money for the designer and end user alike.

REFERENCES

Bullo, F. and Lewis, A.D. (2005), *Geometric Control of Mechanical Systems*, New York, NY: Springer.

Bullo, F. and Lewis, A.D. (2005b), "Low-order Controllability and Kinematic Reductions for Affine Connection Control Systems," *SIAM Journal on Control and Optimization*, Vol 44/3, pp. 885-908.

Bullo, F. and Lynch, K.M. (2001), "Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems", *IEEE Trans. Robotics & Automation.*

Chyba, M., Haberkorn, T., Smith, R. and Choi, S. (2008a), "Autonomous underwater vehicles: Development and Implementation of time and energy efficient trajectories," *Ship Technology Research*, 55/2, pp.36-48.

Chyba, M., Haberkorn, T. and Smith, R. (2008b), "Design and Implementation of Time Efficient Trajectories for Autonomous Underwater Vehicles." *Ocean Engineering*, Ocean Engineering, Vol 35/1, pp. 63-76.

Chyba, M., Haberkorn, T., Smith, R. and Wilkens, G. (2008c), "A Geometric Analysis of Trajectory Design for Underwater Vehicles," *Discrete and Continuous Dynamical Systems*, accepted, to appear 2008.

Chyba, M., and Smith, R.N. (2008), "A First Extension of Geometric Control Theory to Underwater Vehicles," *Proceedings of the 2008 IFAC Workshop on Navigation, Guidance and Control of Underwater Vehicles*, Killaloe, Ireland.

Fossen, Thor I. (1994), *Guidance and Control of Ocean Vehicles.* John Wiley & Sons.

Lewis, A. (2007), "Is it Worth Learning Differential Geometric Methods for Modelling and Control of Mechanical Systems?" *Robotica*, Vol 25/6, pp. 765-777.

Lynch, K.M., Shiroma, N., Arai, H., Tanie, K. (1998), "Motion planning for a 3-DOF robot with a passive joint," *Proceedings of IEEE International Conference on Robotics and Automation*, Leuven, Belgium, pp. 927-932.

Lynch, K.M., Shiroma, N., Arai, H., Tanie, K. (2000), "Collision-free trajectory planning for a 3-DOF robot with a passive joint," *Journal of the Society of Instrument & Control Engineers*, Vol 19/12, pp. 1171-1184.

MIT Sea Grant (2007). "AUV Laboratory at MIT Sea Grant." http://auvlab.mit.edu/vehicles/.

Prestero, T. (2001), "Verification of a Six-Degree of Freedom Simulation Model for the REMUS Autonomous Underwater Vehicle," Master of Science thesis, Massachusetts Institute of Technology and Woods Hole Oceanographic Institution, Joint Program in Applied Ocean Science and Engineering.

Smith, R.N. (2008), "Geometric Control Theory and its Application to Underwater Vehicles," Ph.D. dissertation, Ocean & Resources Engineering Department, University of Hawai'i at Manoa.

Smith, R.N., Chyba, M., Singh, S.B. (2008), "Decoupled Trajectory Planning for a Submerged Rigid Body Subject to Dissipative and Potential Forces," *Proceedings of the IEEE Region 10 Colloquium and Third International Conference on Industrial and Information Systems*, To appear, Kharagpur, India.

Society of Naval Architects and Marine Engineers (SNAME) (1950), "Nomenclature for Treating the Motion of a Submerged Body Through a Fluid," Technical and Research Bulletin, No. 1-5.

Tsukamoto, C.L., Yuh, J., Choi, S.K., Lee, W.C., Lorentz, J. (1999), "Experimental study of advanced controllers for an underwater robotic vehicle thruster system," *International Journal of Intelligent Automation and Soft Computing*, Vol. 5/3, pp. 225-238.