CONSTRUCTING TOPOLOGICALLY DISTINCT ENERGY-CRITICAL CURVES IN THE PATH SPACE OF THE EUCLIDEAN LINE

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ABSTRACT. We construct topologically distinct global, non-embedding solutions to the Euler-Lagrange equation for a natural energy functional on the space of maps $f : \mathbb{R} \to \mathbb{E}$.

1. INTRODUCTION

In [5], Michor and Ratiu defined a natural energy functional (2.3) on the set of curves in the space of embeddings $f : \mathbb{R} \to \mathbb{E}$. The Euler-Lagrange equation (2.4) may be regarded as the geodesic equation for this functional, and so solutions of (2.4) may be thought of as geodesics in the space of maps $f : \mathbb{R} \to \mathbb{E}$. Michor and Ratiu constructed global, C^∞ solutions to (2.4); these solutions all have the property that for each $t \geq 0$, the function $f(-, t)$ is an orientation-preserving embedding (i.e., $f_x > 0$) of $\mathbb{R}$ into $\mathbb{E}$.

The main goal of this paper is the explicit construction of topologically distinct, piecewise C^∞, non-embedding solutions to (2.4). The homotopy classes of these solutions are distinguished by two topological invariants: the Gauss index (5.2) and the Maslov index (5.3) [3]. Our main result (cf. Theorem 7.3) indicates that there exist solutions with arbitrarily prescribed values for these indices.

We will construct these solutions as follows: first, we observe that (2.4) has a large collection of *intermediate differential equations*, as described in [1]. In fact, for any C^∞ function $r(x)$, the first-order PDE

$$f_x^2 f_t = r(x)$$

(cf. (4.1)) is an intermediate equation for (2.4). In §6, we will show how appropriate choices of $r(x)$ and initial data $f(x, 0)$ lead to the construction of local, single-valued solutions for $t \geq 0$. (Compare with [4].) In §7, we will show how these local models may be patched together to construct global solutions of arbitrary topological type.

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2. The energy functional and its Euler-Lagrange equation

This section consists mostly of background material from [5]. Let $\mathbb{E}$ denote the Euclidean line. There is a natural inner product on the space of square-integrable vector fields on $\mathbb{E}$: given

$$V(z) = v(z)\frac{d}{dz}, \qquad W(z) = w(z)\frac{d}{dz},$$

we define

$$\langle V, W\rangle = \int_{\mathbb{E}} v(z)w(z)\,dz. \tag{2.1}$$

Now let $f : \mathbb{R} \to \mathbb{E}$ be a map given by $z = f(x)$ and let $V(x), W(x)$ be vector fields along the map given by

$$V(x) = v(x)\frac{d}{dz}\bigg|_{z=f(x)}, \qquad W(x) = w(x)\frac{d}{dz}\bigg|_{z=f(x)}.$$

Define a symmetric, bilinear pairing $\langle V, W\rangle$ by pulling back the natural inner product (2.1):

$$\langle V, W\rangle = \int_{\mathbb{R}} v(x)w(x)f'(x)\,dx. \tag{2.2}$$

Note that this pairing is not positive-definite unless $f'(x) > 0$ for all $x \in \mathbb{R}$.

Now consider functions $f : \mathbb{R} \times [0, T] \to \mathbb{E}$, thought of as curves γ_f in the infinite-dimensional space $C^\infty(\mathbb{R}, \mathbb{E})$ via

$$\gamma_f(t) = \Big(x \to f(x,t)\Big) \in C^\infty(\mathbb{R}, \mathbb{E}).$$

The "tangent vector" to γ_f at t is the vector field

$$V(x,t) = f_t(x,t)\frac{d}{dz}\bigg|_{z=f(x,t)}$$

along the map $\gamma_f(t)$. It is therefore natural to define the energy functional

$$\mathcal{E}(\gamma_f) = \int_0^T \tfrac{1}{2}\langle V, V\rangle\,dt = \tfrac{1}{2}\int_0^T \int_{\mathbb{R}} f_t^2 f_x\,dx\,dt. \tag{2.3}$$

This energy functional was considered in [5] in the context of Riemannian metrics on the space of *embeddings* $f : \mathbb{R} \to \mathbb{E}$; we wish to generalize to the case of maps $f : \mathbb{R} \to \mathbb{E}$ which are not necessarily embeddings.

We note that this functional is not necessarily finite, so we will consider it as a motivational object. We view it as analogous to the area functional in minimal surface theory: solutions of the Euler-Lagrange equation are interesting, including those for which the original functional fails to be finite.

Suppose that a curve $\gamma_f : [0, T] \to C^\infty(\mathbb{R}, \mathbb{E})$ is stationary for $\mathcal{E}$. Consider a 1-parameter fixed-endpoint variation $\Gamma : [0, T] \times (-\epsilon, \epsilon) \to C^\infty(\mathbb{R}, \mathbb{E})$ with $\Gamma(t, 0) = \gamma_f(t),\ \Gamma(0, s) = \gamma_f(0),\ \Gamma(T, s) = \gamma_f(T)$. Write

$$\Gamma(t, s)(x) = F(x, t, s),$$

with $F(x,t,0) = f(x,t)$, and let $\Gamma_s(t)$ denote $\Gamma(t,s)$. We have

$$\mathcal{E}(\Gamma_s) = \tfrac{1}{2}\int_0^T \int_{\mathbb{R}} F_t^2 F_x \, dx\, dt.$$

Since γ_f is stationary, we have (integrating by parts as usual):

$$\begin{aligned} 0 &= \frac{d}{ds}\Big|_{s=0} \mathcal{E}(\Gamma_s) \\ &= \tfrac{1}{2}\int_0^T \int_{\mathbb{R}} \left(2F_{ts}F_tF_x + F_t^2 F_{xs}\right)\Big|_{s=0} dx\, dt \\ &= -\tfrac{1}{2}\int_0^T \int_{\mathbb{R}} F_s\Big(2(F_tF_x)_t + (F_t^2)_x\Big)\Big|_{s=0} dx\, dt \\ &= -\int_0^T \int_{\mathbb{R}} F_s\Big|_{s=0}\left(2f_tf_{xt} + f_xf_{tt}\right) dx\, dt. \end{aligned}$$

Therefore the natural Euler-Lagrange equation for the energy functional (2.3) is

$$2f_tf_{xt} + f_xf_{tt} = 0. \tag{2.4}$$

We will consider equation (2.4) to be the geodesic equation for the energy functional on the space of curves $f : \mathbb{R} \to \mathbb{E}$. Michor-Ratiu [5] constructed global, non-singular solutions for (2.4) in the case where $f(-,0) : \mathbb{R} \to \mathbb{E}$ is an orientation-preserving embedding; we wish to investigate the global existence of solutions for the larger class of initial conditions where $f(-,0) : \mathbb{R} \to \mathbb{E}$ is an arbitrary C^∞-map.

In fact, we will construct global solutions for the slightly larger class of maps in which the derivative f' of each map $f(-,t_0)$ is allowed to have isolated singularities – specifically, zeros and poles of fractional order. We will denote this class of maps by $\widehat{C}^\infty(\mathbb{R},\mathbb{E})$. (A more precise definition will be given in §3.) Our solutions will have the additional property that the partial derivative f_t – which defines the tangent vector field to the curve γ_f – remains C^∞ except for isolated simple poles, although more general solutions are possible without this restriction. In order to accommodate poles in f', we introduce the notion of the *completed cotangent bundle* in the next section.

3. The completed cotangent bundle

Let $T^*\mathbb{R}$ denote the cotangent bundle of $\mathbb{R}$, with canonical projection $\pi : T^*\mathbb{R} \to \mathbb{R}$. In order to systematically deal with points where $f(x)$ is continuous but the derivative $f'(x)$ is unbounded, we need to "complete" $T^*\mathbb{R}$ by inserting a point at infinity in each π-fiber. We can do this rigorously as follows. Consider a canonical coordinate system (x,p) for $T^*\mathbb{R}$. We take another copy of $T^*\mathbb{R}$ (which we will denote $\overline{T^*\mathbb{R}}$), with canonical coordinates $(x,\bar{p})$, and define

$$\widehat{T}^*\mathbb{R} = T^*\mathbb{R} \cup_\sim \overline{T^*\mathbb{R}},$$

where $(x, p) \sim (x, 1/\bar{p})$ for $p, \bar{p} \neq 0$.

We will let $\pi : \widehat{T}^*\mathbb{R} \to \mathbb{R}$ denote the obvious extension of the canonical projection $\pi : T^*\mathbb{R} \to \mathbb{R}$ to $\widehat{T}^*\mathbb{R}$, and let $\pi_+ : \widehat{T}^*\mathbb{R} \times \mathbb{E}^+ \to \mathbb{R} \times \mathbb{E}^+$ denote the natural extension of π to $\widehat{T}^*\mathbb{R} \times \mathbb{E}^+$; i.e., $\pi_+(x, t, p) = (x, t)$.

We now define two important subsets of $\widehat{T}^*\mathbb{R}$.

Definition 3.1. The *zero section* of $\widehat{T}^*\mathbb{R}$ is the set

$$Z^0 = \{(x, p) \mid p = 0\} \subset \widehat{T}^*\mathbb{R}.$$

The *infinite section* of $\widehat{T}^*\mathbb{R}$ is the set

$$Z^\infty = \{(x, \bar{p}) \mid \bar{p} = 0\} \subset \widehat{T}^*\mathbb{R}.$$

We also set

$$\begin{aligned} Z^0_+ &= Z^0 \times \mathbb{E}^+ \subset \widehat{T}^*\mathbb{R} \times \mathbb{E}^+ \\ Z^\infty_+ &= Z^\infty \times \mathbb{E}^+ \subset \widehat{T}^*\mathbb{R} \times \mathbb{E}^+. \end{aligned}$$

We will consider the following class of maps $f : \mathbb{R} \to \mathbb{E}$:

Definition 3.2. A C^0 function $f : \mathbb{R} \to \mathbb{E}$ is called $\widehat{C}^\infty$ if:

- The derivative $f' : \mathbb{R} \to \widehat{T}^*\mathbb{R}$ is continuous.
- f is C^∞ outside of a discrete set $\mathcal{X} \subset \mathbb{R}$ consisting of all finite (but possibly fractional) order zeros and poles of f'.

We will write $\mathcal{X} = \mathcal{X}^0 \cup \mathcal{X}^\infty$, where

$$\mathcal{X}^0 = \{x \in \mathbb{R} \mid f'(x) = 0\}, \qquad \mathcal{X}^\infty = \{x \in R \mid \frac{1}{f'(x)} = 0\}.$$

We also set

$$\begin{aligned} \mathcal{X}_+ &= \mathcal{X} \times \mathbb{E}^+ \subset \mathbb{R} \times \mathbb{E}^+ \\ \mathcal{X}^0_+ &= \mathcal{X}^0 \times \mathbb{E}^+ \subset \mathbb{R} \times \mathbb{E}^+ \\ \mathcal{X}^\infty_+ &= \mathcal{X}^\infty \times \mathbb{E}^+ \subset \mathbb{R} \times \mathbb{E}^+. \end{aligned}$$

4. Constructing solutions via intermediate equations

Observe that equation (2.4) admits first-order *intermediate differential equations (IDEs)* of the form

$$f_x^2 f_t = r(x) \tag{4.1}$$

for any smooth function $r(x)$. Our solutions will have the property that $r(x) = 0$ precisely at the zeros of f_x; therefore we will use the same notation $\mathcal{X}^0$ to denote the zero set of $r(x)$. Any solution to (4.1) is automatically a solution to (2.4) as well; thus our approach to constructing solutions to (2.4) will be to construct solutions to (4.1), with $r(x)$ chosen in such a way as to obtain various types of singularities in the solution.

We begin by rewriting equation (4.1) as a first-order PDE for the function $p(x,t) = f_x(x,t)$: dividing both sides by f_x^2 and differentiating with respect to x yields:

$$f_{xt} = \left[\frac{r(x)}{f_x^2}\right]_x.$$

We write this as the *reduced differential equation (RDE)* for $p(x,t)$:

$$p_t = \left[\frac{r(x)}{p^2}\right]_x. \tag{4.2}$$

This is an equation of Burgers type. (See [6].) The corresponding equation for $\bar{p}(x,t)$ is

$$\bar{p}_t = -\bar{p}^2[r(x)\bar{p}^2]_x. \tag{4.3}$$

Any solution $p(x,t)$ to (4.2) can be integrated to produce a solution $f(x,t)$ to (4.1) (unique up to an additive constant), which is in turn a solution to (2.4).

We will approach the problem of constructing these solutions via the method of characteristics. At points of the singular set $\mathcal{X}$, the usual initial value problem will be ill-posed, so careful geometric arguments will be required to construct solutions near singular points.

5. Diff$^\infty(\mathbb{R})$-invariance and topological invariants of solutions

Observe that the energy functional (2.3), its Euler-Lagrange equation (2.4), and all the constructions of §3 are invariant under the action of Diff$^\infty(\mathbb{R})$, i.e., under a smooth change of coordinate $x \to \chi(x)$. This follows from the observation that the induced mapping on $\widehat{T}^*\mathbb{R}$,

$$\begin{aligned} (x,p) &\to (\chi(x), p/\chi'(x)) \\ (x,\bar{p}) &\to (\chi(x), \chi'(x)\bar{p}), \end{aligned} \tag{5.1}$$

is a diffeomorphism of $\widehat{T}^*\mathbb{R}$ which preserves the subsets Z^0, Z^∞ and the fibers of the map $\pi : \widehat{T}^*\mathbb{R} \to \mathbb{R}$. Hence there is no preferred Euclidean structure on $\mathbb{R}$, and a solution $f(x,t)$ to (2.4) should be identified with its orbit

$$\{f(\chi(x),t) \mid \chi(x) \in \mathrm{Diff}^\infty(\mathbb{R})\}.$$

The IDE (4.1), however, is not invariant under Diff$^\infty(\mathbb{R})$; it is straightforward to check that under the change of coordinate $x \to \chi(x)$, the function $r(x)$ is transformed to $\frac{r(x)}{(\chi'(x))^2}$.

It is natural to distinguish solutions with the following (symplectic) topological invariants, which are invariant under Diff$^\infty(\mathbb{R})$:

Definition 5.1. Given a $\widehat{C}^\infty$ function $f : \mathbb{R} \to \mathbb{E}$, we define the *Gauss index* $[g]$ and the *Maslov index* $[g']$ of f over a (possibly infinite) interval $[a,b]$ to be

$$[g]f = \frac{1}{\pi} \int_a^b \frac{f''}{1+(f')^2}\, dx = \arctan(f')\big|_b - \arctan(f')\big|_a, \tag{5.2}$$

$$[g']f = \frac{1}{\pi} \int_a^b \frac{f'''}{1+(f'')^2}\, dx = \arctan\,(f'')\big|_b - \arctan\,(f'')\big|_a, \tag{5.3}$$

respectively. If $f(x,t)$ is a $\widehat{C}^\infty$ solution to (2.4), then for any fixed $t \in \mathbb{E}^+$, we define $[g_t]f$ and $[g_t']f$ to be the $[g]$-index and $[g']$-index, respectively, of the function $x \to f(x,t)$.

These quantities count twice the number of counterclockwise rotations in the tangent lines to the images of the graphs

$$\{(x, f(x))\} \subset \mathbb{R} \times \mathbb{E}, \qquad \{(x, f'(x))\} \subset \mathbb{R} \times \widehat{T}^*\mathbb{R}$$

of f and f', respectively. If f is continuous with simple behavior at singular points, then both (5.2) and (5.3) will be well defined. (Note that f'' or f''' may be discontinuous at singular points or at endpoints of the interval, and the arctan function must then be evaluated via one sided limits.) Thus these indices are determined by the singularities and the asymptotic behavior of f. We will construct solutions for which $[g_t]f$ and $[g_t']f$ are integer-valued and constant in t; note that neither of these conditions necessarily holds in general.

Example 5.2. Consider the function

$$f(x) = \int_0^x (\sin u)^{\frac{n}{n+1}}\, du + cx \tag{5.4}$$

where $n \in \mathbb{Z}^+$ is even, $x \in \mathbb{R}$, and $c \in \mathbb{R}$ is chosen so that $f(x)$ is π-periodic. The function

$$p(x) = f'(x) = (\sin x)^{\frac{n}{n+1}} + c$$

is C^∞ except at multiples of π. Moreover,

$$\left(\frac{n+1}{n}\right) f''(x) = \frac{\cos x}{(\sin x)^{\frac{1}{n+1}}}$$

is unbounded at multiples of π, where the image of f' has downward-pointing cusps. (See Figures 1 and 2 for graphs when $n = 2$.) It follows that on the interval $[0,\pi]$, we have $[g]f = 0$ and $[g']f = -1$. By starting with $f(x)$ or $-f(x)$ on $[0, m\pi]$ and smoothly modifying f so that $f'(x)$ is constant on the complement of a slightly larger interval, one can construct $\widehat{C}^\infty$ functions $f(x)$ with $[g]f = 0$ and $[g']f$ an arbitrary integer m. (Non-integer values can be realized by smoothly attaching polynomial or rational functions on the complement.)

Example 5.3. Consider the function

$$f(x) = (\sin x)^{\frac{n}{n+1}}, \tag{5.5}$$

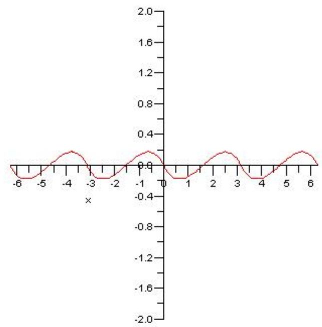

FIGURE 1. Example 5.2: $f(x) = \int_0^x (\sin u)^{2/3}\, du + cx$

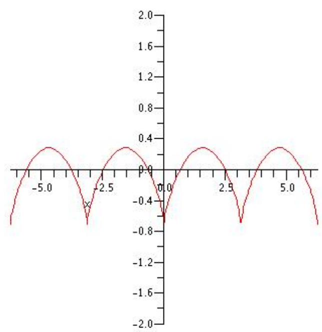

FIGURE 2. Example 5.2: $p(x) = (\sin x)^{2/3} + c$

where $n \in \mathbb{Z}^+$ is even. Here $p(x) = f'(x)$ is unbounded at multiples of π, and it is straightforward to show that on $[0, \pi]$, $[g]f = -1$ and $[g']f = 0$. (See Figures 3 and 4 for graphs when $n = 2$.)

By smoothly attaching combinations of Examples 5.2 and 5.3, one can construct $\widehat{C}^\infty$ functions $f(x)$ with arbitrary Gauss and Maslov indices.

One might attempt to construct solutions of (2.4) of a given topological type simply by taking any RDE (4.2), starting with initial data of the appropriate type constructed from Examples 5.2 and 5.3, and allowing it to evolve. Unfortunately, this initial data may well evolve into discontinuous or multi-valued solutions. In §6, we show how to construct local singular solutions that remain continuous and single-valued for all $t \geq 0$. In §7, we will show how to concatenate these local solutions in order to construct global solutions of arbitrary topological type.

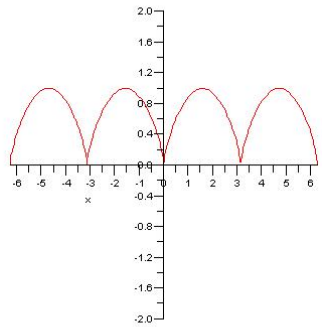

FIGURE 3. Example 5.3: $f(x) = (\sin x)^{2/3}$

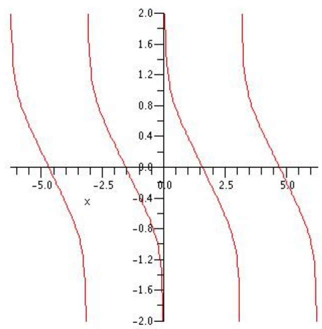

FIGURE 4. Example 5.3: $p(x) = \frac{2}{3}(\sin x)^{-1/3}\cos x$

6. Explicit constructions of local solutions

In this section, we will construct explicit local solutions for (4.2) in a neighborhood of $x = 0$ when $r(x) = \pm x^k$, $k \in \{0, 1, 2, 3\}$. These solutions can then be composed with transformations of the form (5.1) to produce solutions with $r(x) = x^k g(x)$ for any C^∞, nonvanishing function $g(x)$. We will use these local solutions in §7 to construct global solutions from initial data with arbitrary Gauss and Maslov indices.

We begin by examining the Cauchy characteristics for (4.2). The characteristic line field for (4.2) is spanned by the vector field

$$\Gamma_r = 2r(x)\frac{\partial}{\partial x} + p^3\frac{\partial}{\partial t} + pr'(x)\frac{\partial}{\partial p} \tag{6.1}$$

on $T^*\mathbb{R}\times\mathbb{E}^+$. Observe that on the set Z^0_+, this line field is spanned by $r(x)\frac{\partial}{\partial x}$ (except at points of $\pi_+^{-1}(\mathcal{X}^0)\cap Z^0_+$, where $\Gamma_r = 0$). Since the characteristic vector field contains no $\frac{\partial}{\partial t}$ component on Z^0_+, initial data passing through

$Z^0_+ - \pi_+^{-1}(\mathcal{X}^0)$ is ill-posed. Because of this, we will *not* consider solutions which intersect $Z^0_+ - \pi_+^{-1}(\mathcal{X}^0)$. We will, however, construct solutions which intersect Z^0_+ on the *zero curves* $\mathcal{X}^0_+ = \mathcal{X}^0 \times \mathbb{E}^+$, where the characteristic vector field vanishes entirely. In other words, $r(x)$ *must have a zero at any point* x_0 *for which* $p(x_0, t) = 0$ *for any value of* t. (In fact, we will see that if $p(x_0, t) = 0$ for some value of t, then $p(x_0, t) = 0$ for all $t \geq 0$.)

Now consider the characteristic vector field near Z^∞_+. Under the change of coordinates $p \to \bar{p}^{-1}$, the characteristic vector field Γ_r becomes

$$\begin{aligned}\Gamma_r &= 2r(x)\frac{\partial}{\partial x} + \frac{1}{\bar{p}^3}\frac{\partial}{\partial t} - \bar{p}r'(x)\frac{\partial}{\partial \bar{p}} \\ &= \frac{1}{\bar{p}^3}\left(2\bar{p}^3 r(x)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - \bar{p}^4 r'(x)\frac{\partial}{\partial \bar{p}}\right).\end{aligned}$$

It follows that the characteristic line field has a C^∞ extension to $\widehat{T}^*\mathbb{R} \times \mathbb{E}^+ - Z^0_+$, and near Z^∞_+ this extension is spanned by multiples of the vector field

$$\bar{\Gamma}_r = 2\bar{p}^3 r(x)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - \bar{p}^4 r'(x)\frac{\partial}{\partial \bar{p}}. \tag{6.2}$$

Note that on Z^∞_+, the characteristic line field is spanned by $\frac{\partial}{\partial t}$; thus, any characteristic curve which passes through Z^∞_+ at $t = 0$ remains there for all $t \geq 0$. It follows that initial data $\bar{p}(x) = \frac{1}{f_x}(x, 0)$ which intersects Z^∞ at a point x_0 will evolve so as to intersect Z^∞_+ on the causal half ray $\{x_0\} \times \mathbb{E}^+$.

In order to characterize initial data which evolves into a single-valued $\widehat{C}^\infty$-solution, we must understand:

- when the π_+-projections of distinct characteristic curves through an initial arc intersect at a point (x, t) with $t \geq 0$ (thereby creating multivalues in p), and
- how the characteristic curves behave over the singular points $\pi_+^{-1}(\mathcal{X})$.

Regarding the former issue, we will find a simple criterion on the derivative of initial data which ensures that all such intersections occur at points with $t < 0$ (Proposition 6.3). The latter issue, along with the question of how smoothly these evolving arcs extend across $\mathcal{X} \times \mathbb{E}^+$, will be addressed on a case-by-case basis.

6.1. **Case 1.** Let $r(x) = \pm 1$. In this case, (4.2) becomes

$$p_t = \left[\frac{\pm 1}{p^2}\right]_x. \tag{6.3}$$

Since $r(x)$ is nonvanishing, $p(x, t)$ must be nonvanishing as well; thus we will work entirely in $(x, t, \bar{p})$ coordinates. To this end, we will rewrite (6.3) in the form (4.3); i.e.,

$$\bar{p}_t = \mp\tfrac{1}{2}[\bar{p}^4]_x. \tag{6.4}$$

The characteristic line field is spanned by

$$\overline{\Gamma}_r = \pm 2\bar{p}^3 \frac{\partial}{\partial x} + \frac{\partial}{\partial t},$$

where the sign corresponds to the sign of $r(x) = \pm 1$. The characteristic curve passing through the point $(x_0, 0, \bar{p}_0)$ is parametrized by

$$(x(v),\, t(v),\, \bar{p}(v)) = (x_0 \pm 2\bar{p}_0^3 v,\, v,\, \bar{p}_0).$$

Note that these curves are straight lines, and that $\bar{p}$ is constant on each characteristic curve. Given (possibly singular) initial data $\bar{p}_0(x)$, the corresponding solution surface $s : \mathbb{R} \times \mathbb{R}^+ \to \widehat{T}^*\mathbb{R} \times \mathbb{E}^+$ is parametrized by

$$\begin{aligned} x(u,v) &= u \pm 2v\bar{p}_0(u)^3 \\ t(u,v) &= v \\ \bar{p}(u,v) &= \bar{p}_0(u). \end{aligned} \tag{6.5}$$

Now consider the problem of when the π_+-projections of distinct characteristic curves through an initial arc intersect at a point (x, t) with $t \geq 0$. We wish to avoid this situation in order to ensure a single-valued solution for all $t \geq 0$; for this it suffices that the Jacobian determinant $\frac{\partial(x,t)}{\partial(u,v)}$ be nonvanishing for $t \geq 0$ [2]. A straightforward computation shows that

$$\frac{\partial(x,t)}{\partial(u,v)} = 1 \pm 6v\bar{p}_0(u)^2\bar{p}_0'(u).$$

Therefore, it suffices to choose initial data $\bar{p}_0(x)$ such that:

- if $r(x) = 1$, $\bar{p}_0'(x) \geq 0$ whenever $\bar{p}_0'(x)$ is finite;
- if $r(x) = -1$, $\bar{p}_0'(x) \leq 0$ whenever $\bar{p}_0'(x)$ is finite.

(See Figures 5 and 6 for illustrations of "good" and "bad" behavior of characteristics in the (x, t) plane.)

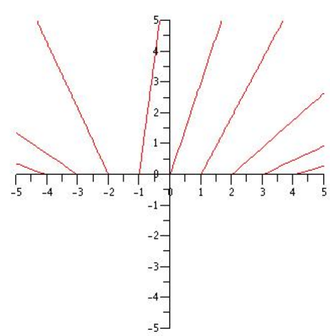

FIGURE 5. "Good" characteristic behavior

We state this as:

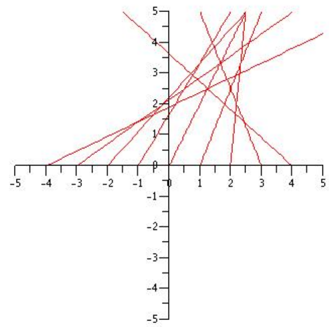

FIGURE 6. "Bad" characteristic behavior

Proposition 6.1. *Given (possibly singular) initial data* $\bar{p}_0(x)$ *for equation* (6.4), *the associated solution* $s : \mathbb{R} \times \mathbb{R}^+ \to \widehat{T}^*\mathbb{R} \times \mathbb{E}^+$ *given by* (6.5) *remains single-valued for all* $t \geq 0$ *if for all* $x \in \mathbb{R}$,

$$\frac{d\bar{p}_0}{dx} \geq 0 \ \left(\text{or equivalently, } \frac{dp_0}{dx} \leq 0\right) \text{ when } r = 1, \tag{6.6}$$

$$\frac{d\bar{p}_0}{dx} \leq 0 \ \left(\text{or equivalently, } \frac{dp_0}{dx} \geq 0\right) \text{ when } r = -1. \tag{6.7}$$

Note that as a consequence of this proposition, $\bar{p}_0(x)$ can have at most one zero; moreover, this zero must be of fractional order less than one in order for $f_0(x)$ to remain finite. If $\bar{p}_0(x_0) = 0$, then the characteristic curve passing through $(x_0, 0, \bar{p}_0(x_0)) = (x_0, 0, 0)$ is parametrized by

$$(x(v), t(v), \bar{p}(v)) = (x_0, v, 0),$$

so the zero in $\bar{p}$ persists at the same x-value for all $t \geq 0$. This corresponds to a cusp in the graph of f, which makes a nontrivial contribution to the Gauss index $[g]f$.

Example 6.2. The initial data $\bar{p}_0(x) = \frac{|x|^{3/2}}{x}$, or equivalently,

$$p_0(x) = \frac{x}{|x|^{3/2}} = \begin{cases} \frac{1}{\sqrt{x}} & x \geq 0 \\ -\frac{1}{\sqrt{-x}} & x < 0 \end{cases},$$

satisfies condition (6.6) with $r = 1$, and so it will yield a global, single-valued solution $\bar{p}(x, t)$ to the RDE

$$\bar{p}_t = -\tfrac{1}{2}[\bar{p}^4]_x.$$

The corresponding solution $p(x, t)$ to the RDE

$$p_t = \left[\frac{1}{p^2}\right]_x$$

has a pole of order $\frac{1}{2}$ at $x = 0$ for all $t \geq 0$. (See Figures 7 and 8 for graphs of $p_0(x) = \frac{x}{|x|^{3/2}}$ and $f_0(x) = 2\sqrt{|x|} - 3$.) Figure 9 shows the π_+-projections

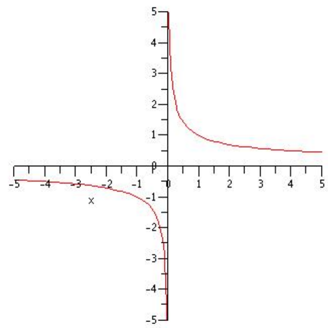

FIGURE 7. Example 6.2: $p_0(x) = \frac{x}{|x|^{3/2}}$

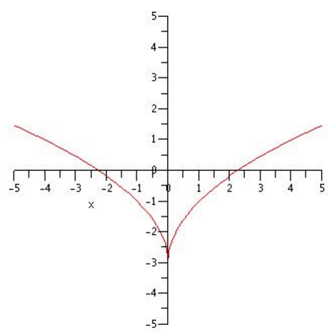

FIGURE 8. Example 6.2: $f_0(x) = 2\sqrt{|x|} - 3$

of the characteristic curves in the (x, t)-plane; these curves do not cross for any $t \geq 0$, and this initial data yields a $\widehat{C}^\infty$, single-valued solution $p(x, t)$ to the RDE

$$p_t = \left[\frac{1}{p^2}\right]_x .$$

6.2. **Focusing and defocusing of characteristics at singularities.** Note that in Figure 9, there is a single, vertical characteristic curve at the singular point $x = 0$, and that nearby characteristic curves diverge from the vertical curve as t increases. For this reason, we call this a *defocusing* singularity. All the remaining examples in this section will have a significantly different characteristic profile at the singular point $x = 0$; see, e.g., Figure 12. These

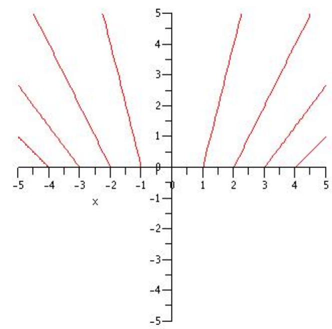

FIGURE 9. Characteristic curves in (x, t)-plane for Example 6.2

will be referred to as *focusing* singularities, and they will be constructed so that $p_0(0) = 0$ (and hence $r(0) = 0$ as well). When we construct global solutions in §7, we will employ initial data with alternating focusing and defocusing singularities.

In order to construct single-valued solutions with this type of singularity, the following conditions must be satisfied:

(1) The function $\frac{r(x)}{p_0(x)^2}$ must remain finite and C^∞ as $x \to 0$ so that the vector field f_t defined by (4.1) will be C^∞ at $x = 0$.
(2) As noted previously, the characteristic vector field will vanish whenever $x = 0$. But in order for the characteristic curves for $x > 0$ and $x < 0$ to patch smoothly along the t-axis as in Figure 12, the characteristic vector field should approach zero *horizontally* as $x \to 0$. Therefore, we must have $\lim_{x \to 0} \left(\frac{p_0(x)^3}{r(x)} \right) = 0$.
(3) In addition to patching smoothly in the (x, t)-plane, the characteristic curves must patch smoothly in the p-coordinate along the t-axis as well. This will require a symmetry condition in the initial data.

Furthermore, we must ensure that the π_+-projections of distinct characteristic curves never intersect at a point (x, t) with $t > 0$. Fortunately, Proposition 6.1 is much broader than it appears: applying a transformation of the form (5.1) (with $\chi'(x) \geq 0$ in order to preserve orientation) transforms $r(x)$ from ± 1 to $\pm(\chi'(x))^2$ and transforms $p(x)$ to $\frac{p(x)}{\chi'(x)}$. Thus we have the following more general statement:

Proposition 6.3. *Given (possibly singular) initial data $p_0(x)$ for equation (4.2), the associated solution remains single-valued for all $t \geq 0$ if for all*

$x \in \mathbb{R}$,

$$\frac{d}{dx}\left(\frac{p_0(x)}{\sqrt{|r(x)|}}\right) \leq 0 \textit{ when } r(x) \geq 0, \tag{6.8}$$

$$\frac{d}{dx}\left(\frac{p_0(x)}{\sqrt{|r(x)|}}\right) \geq 0 \textit{ when } r(x) \leq 0. \tag{6.9}$$

This condition, together with conditions (1) and (2) above, will force $\frac{r(x)}{p_0(x)^2}$ to be a C^∞, nonvanishing function near $x = 0$. We will use all these conditions on our initial data to ensure that solutions remain single-valued for all $t > 0$.

Note that the characteristic vector field (6.1) has the property that

$$\Gamma_r\left(\frac{r(x)}{p^2}\right) = 0;$$

therefore the quantity $\frac{r(x)}{p^2}$ is constant on each characteristic curve. This fact will allow us to integrate the characteristic vector field and to obtain an explicit expression for the characteristic curves in each of our examples below.

6.3. **Case 2.** Let $r(x) = x$. (The case $r(x) = -x$ is similar.) We will construct solutions that intersect the zero locus Z_+^0 at $x = 0$; to this end, we will work in (x, t, p) coordinates.

In this case, (4.2) becomes

$$p_t = \left[\frac{x}{p^2}\right]_x. \tag{6.10}$$

The characteristic line field is spanned by

$$\Gamma_r = 2x\frac{\partial}{\partial x} + p^3\frac{\partial}{\partial t} + p\frac{\partial}{\partial p}.$$

Consider the characteristic curve passing through a point $(x_0, 0, p_0)$ with $x_0, p_0 \neq 0$. Near such a point, we can write

$$\Gamma_r \sim \frac{\partial}{\partial x} + \frac{p^3}{2x}\frac{\partial}{\partial t} + \frac{p}{2x}\frac{\partial}{\partial p}.$$

This vector field can be integrated directly, and the characteristic curve is parametrized by

$$(x,\, t(x),\, p(x)) = (x,\, \tfrac{1}{3}a_0^3(|x|^{3/2} - |x_0|^{3/2}),\, a_0|x|^{1/2}), \tag{6.11}$$

where $a_0 = \frac{p_0}{|x_0|^{1/2}}$. The π_+-projection of this curve is the cubic curve

$$\left(3t + p_0^3\right)^2 = a_0^3|x|^3. \tag{6.12}$$

If $a_0 < 0$ – which corresponds to a choice of sign in the initial data $p_0(x)$ – then this curve will intersect the t-axis at some point $t > 0$, thereby creating a focusing singularity. Since $\frac{x}{p^2} = \frac{\pm 1}{a_0^2}$ is constant on this curve, we must have $p(0,t) = 0$ for all $t > 0$. So in order to construct the desired solutions, we need to ensure that:

- The derivative condition of Proposition 6.3 is satisfied. This can be achieved by choosing $p_0(x) = |x|^{1/2}P(x)$, where P is a nonvanishing function which is decreasing when $x \geq 0$ and increasing when $x \leq 0$. Moreover, since we want $\frac{p}{|x|^{1/2}} < 0$, we should choose P to be a negative-valued function.
- The characteristic curves patch together continuously when $x = 0$. This can be arranged by choosing $P(x)$ so that $P(-x) = P(x)$, for then $p_0(-x_0) = p_0(x_0)$, and the characteristic curves through the points $(x_0, 0, p_0)$ and $(-x_0, 0, p_0)$ are two branches of the same cubic curve (6.12). The resulting $\widehat{C}^\infty$ solution $p(x,t)$ has a cusp singularity along the t-axis, which makes a nontrivial contribution to the Maslov index $[g']f$.

Example 6.4. Take as initial data $p_0(x) = -|x|^{1/2}(1 + \frac{1}{100}x^2)$. This corresponds to $f_0(x) = -\frac{\text{sgn}(x)}{1050}|x|^{3/2}(3x^2 + 700) + c$. (See Figures 10 and 11 for graphs of $p_0(x)$ and $f_0(x)$.) Figure 12 shows the π_+-projections of the

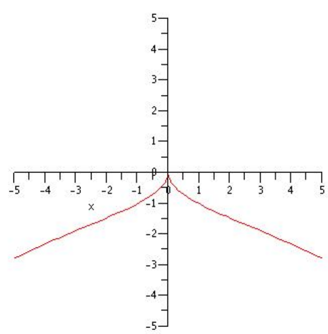

FIGURE 10. Example 6.4: $p_0(x) = -|x|^{1/2}(1 + \frac{1}{100}x^2)$

characteristic curves in the (x,t)-plane; these curves do not cross for any $t \geq 0$ (except where they meet along the t-axis), and this initial data yields a global, single-valued solution $p(x,t)$ to the RDE

$$p_t = \left[\frac{x}{p^2}\right]_x .$$

The corresponding solution $f(x,t)$ has Gauss and Maslov indices $[g_0]f = 0$ and $[g_0']f = 1$, respectively, over the entire real line at $t = 0$.

6.4. **Case 3.** Let $r(x) = x^2$. (The case $r(x) = -x^2$ is similar.) We will construct solutions that intersect the zero locus Z_+^0 at $x = 0$; to this end, we will work in (x,t,p) coordinates.

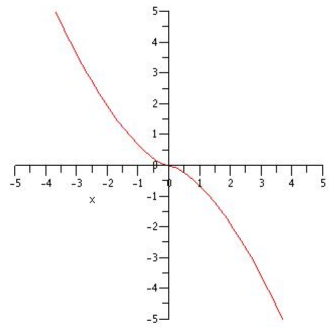

FIGURE 11. Example 6.4: $f_0(x) = -\frac{\operatorname{sgn}(x)}{1050}|x|^{3/2}(3x^2 + 700)$

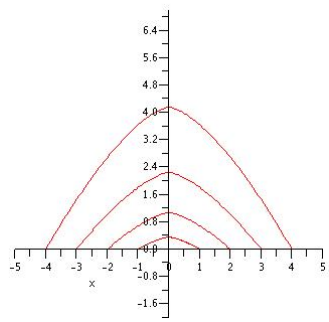

FIGURE 12. Characteristic curves for Example 6.4

In this case, (4.2) becomes

$$p_t = \left[\frac{x^2}{p^2}\right]_x . \tag{6.13}$$

The characteristic line field is spanned by

$$\Gamma_r = 2x^2\frac{\partial}{\partial x} + p^3\frac{\partial}{\partial t} + 2xp\frac{\partial}{\partial p}.$$

Consider the characteristic curve passing through a point $(x_0, 0, p_0)$ with $x_0, p_0 \neq 0$. Near such a point, we can write

$$\Gamma_r \sim \frac{\partial}{\partial x} + \frac{p^3}{2x^2}\frac{\partial}{\partial t} + \frac{p}{x}\frac{\partial}{\partial p}.$$

This vector field can be integrated directly, and the characteristic curve is parametrized by

$$(x,\, t(x),\, p(x)) = (x,\, \tfrac{1}{4}a_0^3(x^2 - x_0^2),\, a_0 x), \tag{6.14}$$

where $a_0 = \frac{p_0}{x_0}$. The π_+-projection of this curve is the parabola

$$t = \tfrac{1}{4}a_0^3(x^2 - x_0^2). \tag{6.15}$$

If $a_0 < 0$ – which corresponds to a choice of sign in the initial data $p_0(x)$ – then this parabola will intersect the t-axis at some point $t > 0$, thereby creating a focusing singularity. Since $\frac{x^2}{p^2} = \frac{1}{a_0^2}$ is constant on this curve, we must have $p(0,t) = 0$ for all $t > 0$. So in order to construct the desired solutions, we need to ensure that:

- The derivative condition of Proposition 6.3 is satisfied. This can be achieved by choosing $p_0(x) = xP(x)$, , where P is a nonvanishing function which is decreasing when $x \geq 0$ and increasing when $x \leq 0$. Moreover, since we want $\frac{p}{x} < 0$, we should choose P to be a negative-valued function.
- The characteristic curves patch together smoothly when $x = 0$. This can be arranged by choosing $P(x)$ so that $P(-x) = P(x)$, for then $p_0(-x_0) = -p_0(x_0)$, and the characteristic curves through the points $(x_0, 0, p_0)$ and $(-x_0, 0, -p_0)$ are the two branches of the same parabolic curve (6.15). The resulting $\widehat{C}^\infty$ solution $p(x,t)$ is in fact C^∞ along the t-axis.

Example 6.5. Take as initial data $p_0(x) = -x(1+\frac{1}{100}x^2)$. This corresponds to $f_0(x) = -(\frac{1}{2}x^2+\frac{1}{400}x^4)+c$. (See Figures 13 and 14 for graphs of $p_0(x)$ and $f_0(x)$.) Figure 15 shows the π_+-projections of the characteristic curves in

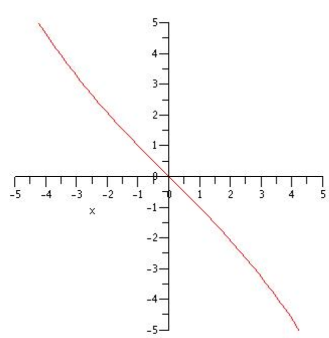

FIGURE 13. Example 6.5: $p_0(x) = -x(1 + \frac{1}{100}x^2)$

the (x,t)-plane; these curves do not cross for any $t \geq 0$ (except where they meet along the t-axis), and this initial data yields a global, single-valued solution $p(x,t)$ to the RDE

$$p_t = \left[\frac{x^2}{p^2}\right]_x.$$

The corresponding solution $f(x,t)$ has Gauss and Maslov indices $[g_0]f = -1$

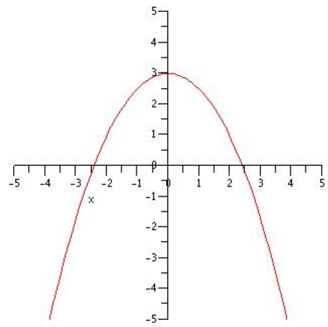

FIGURE 14. Example 6.5: $f_0(x) = -(\frac{1}{2}x^2 + \frac{1}{400}x^4) + 3$

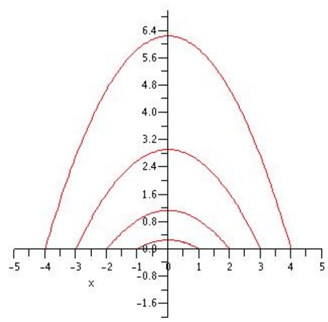

FIGURE 15. Characteristic curves for Example 6.5

and $[g_0']f = 0$, respectively, over the entire real line at $t = 0$.

6.5. **Case 4.** Let $r(x) = x^3$. (The case $r(x) = -x^3$ is similar.) We will construct solutions that intersect the zero locus Z_+^0 at $x = 0$; to this end, we will work in (x, t, p) coordinates.

In this case, (4.2) becomes

$$p_t = \left[\frac{x^3}{p^2}\right]_x. \tag{6.16}$$

The characteristic line field is spanned by

$$\Gamma_r = 2x^3\frac{\partial}{\partial x} + p^3\frac{\partial}{\partial t} + 3px^2\frac{\partial}{\partial p}.$$

Consider the characteristic curve passing through a point $(x_0, 0, p_0)$ with $x_0, p_0 \neq 0$. Near such a point, we can write

$$\Gamma_r \sim \frac{\partial}{\partial x} + \frac{p^3}{2x^3}\frac{\partial}{\partial t} + \frac{3p}{2x}\frac{\partial}{\partial p}.$$

This vector field can be integrated directly, and the characteristic curve is parametrized by

$$(x,\, t(x),\, p(x)) = (x,\, \tfrac{1}{5}a_0^3(|x|^{5/2} - |x_0|^{5/2}),\, a_0|x|^{3/2}), \tag{6.17}$$

where $a_0 = \frac{p_0}{|x_0|^{3/2}}$. The π_+-projection of this curve is the quintic curve

$$\left(5t + \frac{p_0^3}{x_0^2}\right)^2 = a_0^3|x|^5. \tag{6.18}$$

If $a_0 < 0$ – which corresponds to a choice of sign in the initial data $p_0(x)$ – then this curve will intersect the t-axis at some point $t > 0$, thereby creating a focusing singularity. Since $\frac{x^3}{p^2} = \frac{\pm 1}{a_0^2}$ is constant on this curve, we must have $p(0,t) = 0$ for all $t > 0$. So in order to construct the desired solutions, we need to ensure that:

- The derivative condition of Proposition 6.3 is satisfied. This can be achieved by choosing $p_0(x) = |x|^{3/2}P(x)$, where P is a nonvanishing function which is decreasing when $x \geq 0$ and increasing when $x \leq 0$. Moreover, since we want $\frac{p}{|x|^{3/2}} < 0$, we should choose P to be a negative-valued function.
- The characteristic curves patch together continuously when $x = 0$. This can be arranged by choosing $P(x)$ so that $P(-x) = P(x)$, for then $p_0(-x_0) = p_0(x_0)$, and the characteristic curves through the points $(x_0, 0, p_0)$ and $(-x_0, 0, p_0)$ are two branches of the same quintic curve (6.18). The resulting $\widehat{C}^\infty$ solution $p(x,t)$ is C^1 along the t-axis.

Example 6.6. Take as initial data $p_0(x) = -|x|^{3/2}(1 + \frac{1}{100}x^2)$. This corresponds to $f_0(x) = -\frac{\operatorname{sgn}(x)}{450}|x|^{5/2}(x^2 + 180) + c$. (See Figures 16 and 17 for graphs of $p_0(x)$ and $f_0(x)$.) Figure 18 shows the π_+-projections of the

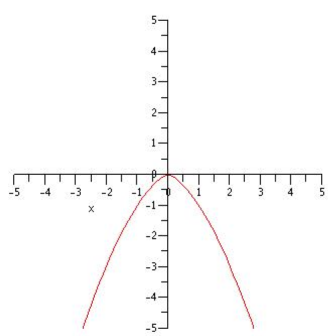

FIGURE 16. Example 6.6: $p_0(x) = -|x|^{3/2}(1 + \frac{1}{100}x^2)$

characteristic curves in the (x,t)-plane; these curves do not cross for any

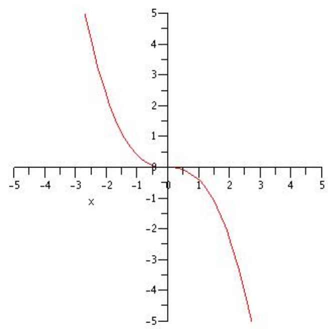

FIGURE 17. Example 6.6: $f_0(x) = -\frac{\operatorname{sgn}(x)}{450}|x|^{5/2}(x^2 + 180)$

$t \geq 0$ (except where they meet along the t-axis), and this initial data yields a global, single-valued solution $p(x,t)$ to the RDE

$$p_t = \left[\frac{x^3}{p^2}\right]_x .$$

The corresponding solution $f(x,t)$ has Gauss and Maslov indices $[g_0]f = 0$

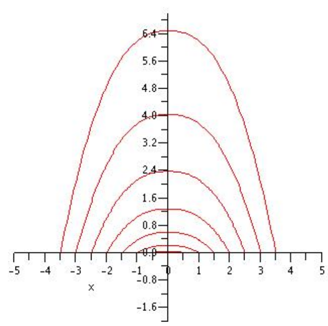

FIGURE 18. Characteristic curves for Example 6.6

and $[g_0']f = -1$, respectively, over the entire real line at $t = 0$.

7. Assembling global solutions

We can use the results of §6 to assemble initial data $p_0(x)$ which evolves into a global $\widehat{C}^\infty$ solution $p(x,t)$ to (4.2) with arbitrary $[g_t]$, $[g_t']$-indices. We do this by first choosing initial data $p_0(x)$ (or $\bar{p}_0(x)$, as appropriate) with the desired topological properties. We then choose $r(x)$ with an appropriate singular set $\mathcal{X}$, with each singularity based on one of the local models of §6. In order to ensure that the solution to (4.2) exists and remains single-valued for all $t > 0$, we must arrange that the π_+-projections of the characteristics to $\mathbb{R} \times \mathbb{E}^+$ fill out all of $\mathbb{R} \times \mathbb{E}^+$ without crossing. We accomplish this by placing the singularities of $p_0(x)$ so that they alternate between focusing and defocusing type. (See Figure 21 for an illustration.) As in the local examples of §6, we must ensure that:

- The derivative condition of Proposition 6.3 is satisfied.
- In the case of a focusing singularity, a symmetry condition holds on the initial data so that the characteristic curves patch continuously across the singularity.

Then we must smoothly concatenate this initial data over $\mathbb{R} - \mathcal{X}$ so that these conditions are respected.

7.1. **Solutions with arbitrary Gauss index.** We can combine Cases 1 and 3 of §6 to construct solutions whose Maslov index $[g_t']$ is zero and whose Gauss index $[g_t]$ is an arbitrary negative integer; positive integer values may be obtained by reversing the signs of $r(x)$ and $p_0(x)$.

Example 7.1. Let $r_0(x)$ be the continuous, piecewise smooth function defined by

$$r_0(x) = \begin{cases} \frac{1}{2}\left(1 - \cos(\frac{\pi}{2}x)\right) & |x| \leq 4 \\ 0 & |x| > 4 \end{cases}.$$

(Choosing $r_0(x) = 0$ for $|x| \geq 4$ will allow us to construct examples with nice asymptotic behavior.) The corners of $r_0(x)$ can be smoothed out to obtain a C^∞ approximation $r(x)$ of $r_0(x)$, as shown in Figure 19. Similarly,

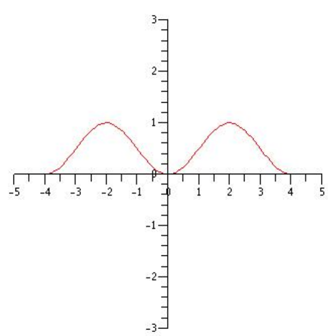

FIGURE 19. Example 7.1: $r(x)$

we can smoothly concatenate scaled and translated copies of the initial data functions from Examples 6.2 and 6.5 in order to create a global initial data function $p_0(x)$ which is C^∞ away from its poles, has a zero of order 1 at each quadratic zero of $r(x)$, and has a pole of order $\frac{1}{2}$ between each pair of zeros of $r(x)$. Since the derivative conditions of Proposition 6.3 are open conditions, this construction can be performed so as to preserve these conditions. We can also impose asymptotic conditions, e.g., that $\lim_{x\to\pm\infty} p_0(x) = 0$. An example of such a function $p_0(x)$ and the corresponding $f_0(x)$ are shown in Figure 20. This initial data has $[g]f_0 = -2$, $[g']f_0 = 0$; it should be clear how to iterate this construction to produce a function $r(x)$ and initial data $p_0(x)$ whose Gauss index is an arbitrary negative integer. Figure 21 shows

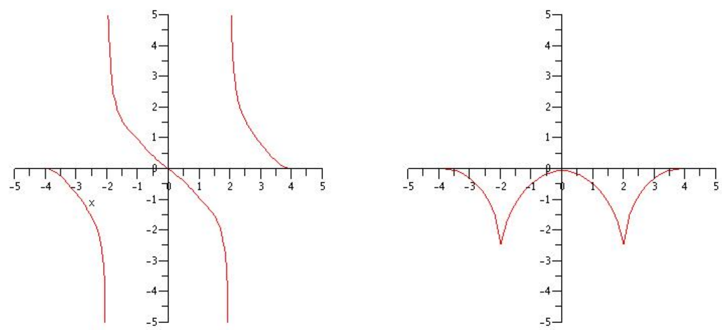

FIGURE 20. Example 7.1: Graphs of $p_0(x)$ and $f_0(x)$

the π_+-projections of the characteristic curves corresponding to this initial data in the (x,t)-plane; these curves do not cross for any $t > 0$ (except where they meet along the t-axis), and this initial data yields a global, $\widehat{C}^\infty$ solution $p(x,t)$ to the RDE

$$p_t = \left[\frac{r(x)}{p^2}\right]_x$$

for $|x| < 4$. The characteristic vector field vanishes for $|x| \geq 4$, but this solution may be C^∞-extended to $\mathbb{R}\times\mathbb{E}^+$ by defining $p(x,t) = 0$ for $|x| \geq 4$. Figures 22 and 23 illustrate the evolution of this initial data at $t = 0$, $t = 1$,

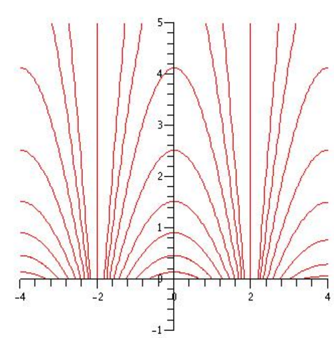

FIGURE 21. Characteristic curves for Example 7.1

and $t = 2$. It is clear that this solution has $[g_t] = -2$, $[g_t'] = 0$ for all $t \geq 0$.

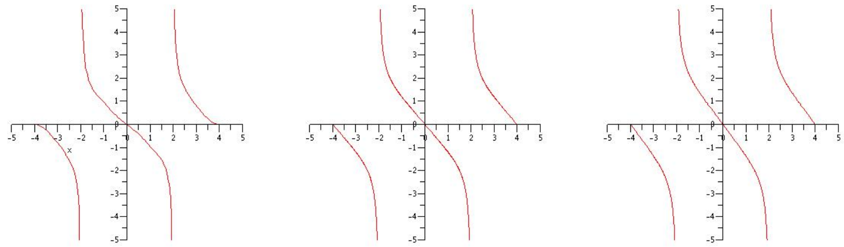

FIGURE 22. Example 7.1: Graphs of $x \to p(x,t)$ for $t = 0, 1, 2$

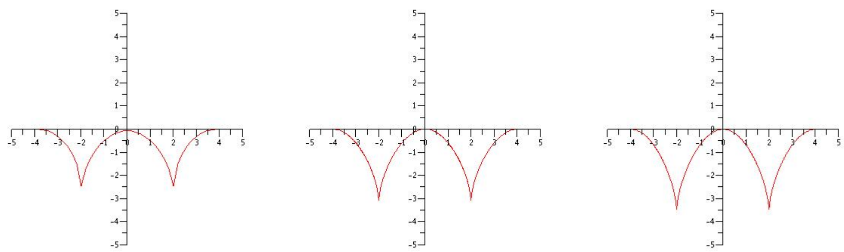

FIGURE 23. Example 7.1: Graphs of $x \to f(x,t)$ for $t = 0, 1, 2$

7.2. **Solutions with arbitrary Maslov index.** We can combine Cases 1, 2, and 4 of §6 to construct solutions whose Gauss index $[g_t]$ is zero and whose Maslov index $[g_t']$ is an arbitrary positive integer; negative integer values may be obtained by reversing the signs of $r(x)$ and $p_0(x)$.

Example 7.2. Let $r_0(x)$ be the continuous, piecewise smooth function defined by

$$r_0(x) = \begin{cases} \cos(\frac{\pi}{2}x) & -2 \le x \le 0 \ \text{ or } \ 2 \le x \le 4 \\ \cos^3(\frac{\pi}{2}x) & -5 \le x \le -2 \ \text{ or } \ 0 \le x \le 2 \ \text{ or } \ 4 \le x \le 5 \\ 0 & |x| > 5 \end{cases} .$$

(Choosing $r_0(x) = 0$ for $|x| \ge 5$ will allow us to construct examples with nice asymptotic behavior.) $r_0(x)$ can be smoothed out to obtain a C^∞ approximation $r(x)$ of $r_0(x)$, as shown in Figure 24. This function has zeros of order 1 at $x = -1$ and $x = 3$, and zeros of order 3 at $x = -3$ and $x = 1$. We can smoothly concatenate scaled and translated copies of the initial data functions from Examples 6.2, 6.4, and 6.6 (with the signs reversed in Example 6.6) in order to create a global initial data function $p_0(x)$ which is C^∞ away from its poles and zeros, has a zero of order $\frac{1}{2}$ at each linear zero of $r(x)$ and a zero of order $\frac{3}{2}$ at each cubic zero of $r(x)$, and has a pole of order $\frac{1}{2}$ between each pair of zeros of $r(x)$. Since the derivative conditions of Proposition 6.3 are open conditions, this construction can be

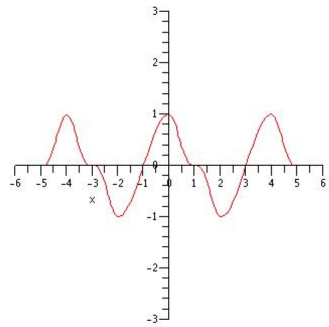

FIGURE 24. Example 7.2: $r(x)$

performed so as to preserve these conditions. We can also impose asymptotic conditions, e.g., that $\lim_{x\to\pm\infty} p_0(x) = 0$. An example of such a function $p_0(x)$ and the corresponding $f_0(x)$ are shown in Figure 25. This initial data has $[g]f_0 = 0,\ [g']f_0 = 2$; it should be clear how to iterate this construction to produce a function $r(x)$ and initial data $p_0(x)$ whose Maslov index is an arbitrary positive integer. Figure 26 shows the π_+-projections of the

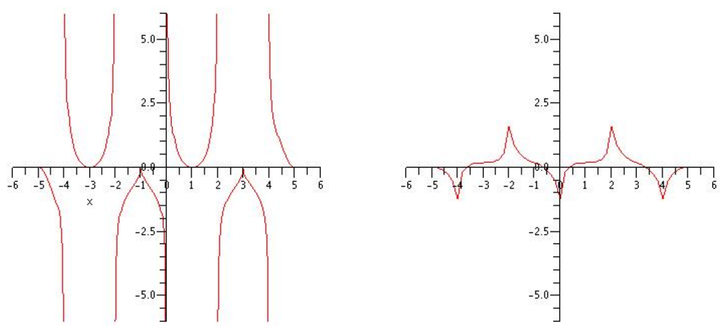

FIGURE 25. Example 7.2: Graphs of $p_0(x)$ and $f_0(x)$

characteristic curves corresponding to this initial data in the (x,t)-plane; these curves do not cross for any $t > 0$ (except where they meet along the t-axis), and this initial data yields a global, $\widehat{C}^\infty$ solution $p(x,t)$ to the RDE

$$p_t = \left[\frac{r(x)}{p^2}\right]_x$$

for $|x| < 5$. The characteristic vector field vanishes for $|x| \geq 5$, but this solution may be C^∞-extended to $\mathbb{R}\times\mathbb{E}^+$ by defining $p(x,t) = 0$ for $|x| \geq 5$. Figures 27 and 28 illustrate the evolution of this initial data at $t = 0$, $t = 1$, and $t = 2$. It is clear that this solution has $[g_t] = 0,\ [g'_t] = 2$ for all $t \geq 0$.

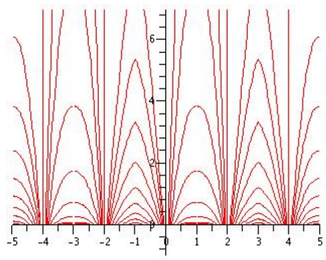

FIGURE 26. Characteristic curves for Example 7.2

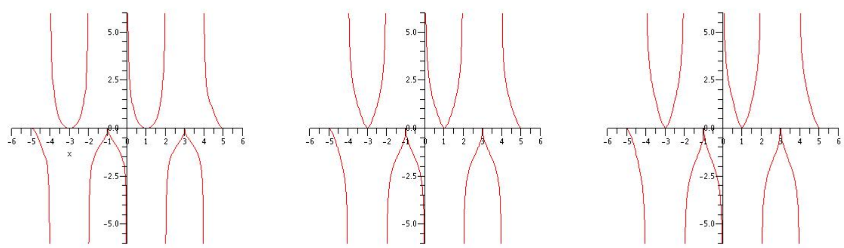

FIGURE 27. Example 7.2: Graphs of $x \to p(x,t)$ for $t = 0, 1, 2$

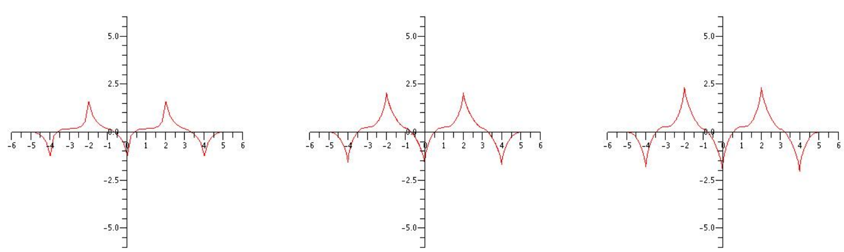

FIGURE 28. Example 7.2: Graphs of $x \to f(x,t)$ for $t = 0, 1, 2$

Because these solutions are identically zero outside of a compact set, it is clear that they may be concatenated so as to create solutions with arbitrarily prescribed values for the Gauss and Maslov indices. Therefore, we have the following theorem:

Theorem 7.3. *Given any two integers m, n, there exists a $\widehat{C}^\infty$ solution $f(x,t)$, defined for $t \geq 0$, of (2.4) with $[g]f = m$, $[g']f = n$.*

8. Closing remarks

While Examples 7.1 and 7.2 are fairly specific, it is possible to construct global solutions to (2.4) from a wide range of initial data. Using our methods, local solutions with focusing singularities may be constructed with $r = x^k$ for any positive integer k, and the $\text{Diff}^\infty(\mathbb{R})$-invariance of solutions provides a wide array of possible initial conditions $p_0(x)$. The only conditions that must be met are the open conditions described in §6: the derivative condition of Proposition 6.3, and a symmetry condition to ensure continuous patching of the characteristic curves across the singularity. It is also possible to weaken the smoothness assumption on f_t, which increases the possible types of singularities that may occur in $p_0(x)$.

Defocusing singularities, on the other hand, appear to be more elusive. We have been unable to construct any defocusing singularities other than those based on the local model in Case 1 of §6. The obstruction is that for any choice of $r(x)$ with a zero – or, for that matter, a pole – at $x = 0$ and initial data $p_0(x)$ of the appropriate sign to create a defocusing singularity, the π_+-projections of the characteristic curves fail to fill out the entire (x, t)-plane for $t > 0$, and therefore solutions fail to exist in any neighborhood of $x = 0$ for any $t > 0$. A typical profile of these characteristics is shown in Figure 29.

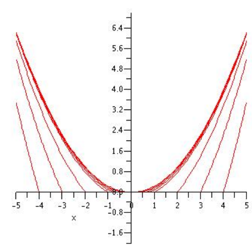

Figure 29. Characteristic curves that fail to fill out the (x, t)-plane

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