SECOND-ORDER TYPE-CHANGING EVOLUTION EQUATIONS WITH FIRST-ORDER INTERMEDIATE EQUATIONS

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In memory of Robert B. Gardner.

ABSTRACT. This paper presents a partial classification for $\mathcal{C}^\infty$ type-changing symplectic Monge-Ampère partial differential equations (PDEs) that possess an infinite set of first-order intermediate PDEs. The normal forms will be quasi-linear evolution equations whose types change from hyperbolic to either parabolic or to zero. The zero points can be viewed as analogous to singular points in ordinary differential equations. In some cases, intermediate PDEs can be used to establish existence of solutions for ill-posed initial value problems.

1. INTRODUCTION

A first-order intermediate partial differential equation (PDE) for a second-order Monge-Ampère PDE (2.3) is defined by the property that every solution of the former is also a solution of the latter. The objective of this paper is to use the set of intermediate PDEs, viewed as an invariant, in the local classification of $\mathcal{C}^\infty$-symplectic Monge-Ampère PDEs with type-changing singularities. This classification will be up to symplectomorphism (i.e., $\mathcal{C}^\infty$-symplectic changes of coordinates on the underlying cotangent space), and the PDEs considered will have infinite sets of intermediate equations.

The main results consist of a collection of invariants and local normal forms for involutive type-changing symplectic Monge-Ampère PDEs with specific sets of intermediate PDEs. The normal forms will be quasi-linear evolution PDEs which change type from hyperbolic to parabolic, or to zero, where the PDE completely degenerates (cf. §7). At the zero points, standard existence techniques cannot be applied to the natural initial value problem. However, in some cases an intermediate PDE can be used to "repair" such an ill-posed second-order problem by replacing it with a well-posed first-order problem. Thus, the main motivation for this paper is to provide a

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local $\mathcal{C}^\infty$-existence theory for type-changing Monge-Ampère PDE that are symplectomorphic to the normal forms (cf. §8).

Moreover, invariant structural details can be inferred from the normal forms. For example: symmetries of a $\mathcal{C}^\infty$ type-changing symplectic Monge-Ampère equation need not extend smoothly to the zero locus; additionally, there exist distinguished solution foliations intersecting the zero locus.

Theorem 7.7 indicates that one of our classes of PDEs with type-changing singularities consists of equations that are symplectomorphic to Euler-Lagrange PDEs for indefinite Lagrangians of the form $f_x L(f_t)$ (cf. §8.1). As an example, Michor and Ratiu introduced the Lagrangian $\frac{1}{2} f_x f_t^2$ as an energy functional on the path space of the Euclidean line (i.e., a natural Riemannian metric on the space of $\mathcal{C}^2$-mappings of $\mathbb{R}$ into $\mathbb{E}$ [Mar70] [MR98]). In [CKW] we will use intermediate PDEs and "repaired" initial data to construct topologically distinct global solutions for this Euler-Lagrange PDE. Global topology will compel the initial data to interact with the zero locus of the PDE.

Historically, intermediate PDEs arose as a 19th-century method for constructing "general solutions" for nonlinear PDEs. Lie and Darboux first used the set of intermediate PDEs as an invariant to characterize the wave equation up to local symplectomorphism (Theorem 6.6). We will refine this method (cf. §4, §5, §6). The main results appear in §7, with consequences discussed in §8.

A more subtle classification scheme for more general hyperbolic or parabolic PDEs, which uses the collection of higher order conservation laws, can be found in [BG95a], [BG95b], [Cle97a], [Cle97b]. Darboux integrals represent another generalization of intermediate PDEs [Jur96], [JA97]. Contact singularities in intermediate PDEs for hyperbolic PDEs were studied in [Cle00]. The normal forms of Theorems 6.4 and 7.14 are related to the normal forms of [BG95b] and [Mar70], respectively. We note in passing that our definition of an involutive zero locus is a refinement of Cartan's definition of "singular solution" for an exterior differential system [Car45].

2. Symplectic Monge-Ampère PDEs

The material in this section is well-known; it originated with Lepage [Lep50] and was developed in detail in [Lyc79] and [LRC93].

Let (M, ω) be a symplectic 4-manifold M with symplectic form ω. In local symplectic coordinates (x, y, p, q) on M, we have $\omega = dp \wedge dx + dq \wedge dy$. M is locally equivalent to the cotangent bundle $T^*\mathbb{R}^2$, and as such, M carries a locally defined 1-form $\theta = p\,dx + q\,dy$ with the property that $d\theta = \omega$.

Definition 2.1. A *symplectic Monge-Ampère PDE* is given by a symplectic 4-manifold (M, ω) and another 2-form Ω on M. A *(generalized) solution* for the PDE is given by a 2-dimensional submanifold $s : D \subset \mathbb{R}^2 \to M$ such that

$$s^*\omega = s^*\Omega = 0.$$

By adding a suitable multiple of ω to Ω, we may assume that $\Omega \wedge \omega = 0$, and we will do so henceforth. A 2-form Ω with this property is called *effective*.

In terms of the local symplectic coordinates defined above, we can then write

$$\begin{aligned}(2.1)\quad \Omega = E\,dp \wedge dq + A\,dp \wedge dy \\ + B\,(dq \wedge dy - dp \wedge dx) + C\,dx \wedge dq + D\,dx \wedge dy,\end{aligned}$$

where A, B, C, D, E are functions of (x, y, p, q). Ω represents the PDE

$$E\,(f_{xx}\,f_{yy} - f_{xy}^2) + A\,f_{xx} + 2B\,f_{xy} + C\,f_{yy} + D = 0, \tag{2.2}$$

where the coefficients are functions of the four variables (x, y, f_x, f_y). Two symplectic Monge-Ampère PDEs $(M_1, \omega_1, \Omega_1)$, $(M_2, \omega_2, \Omega_2)$ are *locally symplectically equivalent* at $e_1 \in M_1$, $e_2 \in M_2$ if there exists a locally defined $\mathcal{C}^\infty$-symplectomorphism $\Psi : M_1 \to M_2$ such that $\Psi(e_1) = e_2$ and

$$\Psi^*\,\Omega_2 \equiv m\,\Omega_1 \mod \omega_1$$

for some locally defined, nonvanishing function $m : M_1 \to \mathbb{R}$.

Contrast this setup with that for a general Monge-Ampère PDE: in the general case, let $(\widetilde{M}, \tilde{\theta})$ be a contact 5-manifold with contact form $\tilde{\theta}$. In local contact coordinates (x, y, z, p, q) on $\widetilde{M}$, we have $\tilde{\theta} = dz - p\,dx - q\,dy$, and $\widetilde{M}$ is locally equivalent to $J^1(\mathbb{R}^2, \mathbb{R})$.

Definition 2.2. A *Monge-Ampère PDE* is given by a contact 5-manifold $(\widetilde{M}, \tilde{\theta})$ and an effective 2-form $\widetilde{\Omega}$ on $\widetilde{M}$. A *(generalized) solution* for the PDE is given by a 2-dimensional submanifold $\tilde{s} : D \subset \mathbb{R}^2 \to \widetilde{M}$ such that

$$\tilde{s}^*\tilde{\theta} = \tilde{s}^*d\tilde{\theta} = \tilde{s}^*\widetilde{\Omega} = 0.$$

As in the previous case, $\widetilde{\Omega}$ represents the PDE

$$E\,(f_{xx}\,f_{yy} - f_{xy}^2) + A\,f_{xx} + 2B\,f_{xy} + C\,f_{yy} + D = 0, \tag{2.3}$$

where now the coefficients are now functions of the five variables $(x, y, z, p, q) = (x, y, f, f_x, f_y)$. Two Monge-Ampère PDEs $(\widetilde{M}_1, \tilde{\theta}_1, \widetilde{\Omega}_1,)$, $(\widetilde{M}_2, \tilde{\theta}_2, \widetilde{\Omega}_2)$ are *locally contact equivalent* at $\tilde{e}_1 \in \widetilde{M}_1$, $\tilde{e}_2 \in \widetilde{M}_2$ if there exists a locally defined $\mathcal{C}^\infty$-contact transformation $\tilde{\Psi} : \widetilde{M}_1 \to \widetilde{M}_2$ such that $\tilde{\Psi}(\tilde{e}_1) = \tilde{e}_2$ and

$$\tilde{\Psi}^*\,\widetilde{\Omega}_2 \equiv \tilde{m}\,\widetilde{\Omega}_1 \mod \tilde{\theta}_1, d\tilde{\theta}_1$$

for some locally defined, nonvanishing function $\tilde{m} : \widetilde{M} \to \mathbb{R}$.

Note that a symplectic Monge-Ampère PDE may be "partially prolonged" to a general Monge-Ampère PDE in a straightforward way via a projection $\rho : \widetilde{M} \to M$ satisfying the condition that $d\tilde{\theta} = -\rho^*(\omega)$. (In this case, (M, ω, Ω) is called a *symplectic reduction* of $(\widetilde{M}, \tilde{\theta}, \widetilde{\Omega})$.) In local coordinates, ρ is given by $\rho(x, y, z, p, q) = (x, y, p, q)$; thus, we see that a symplectic Monge-Ampère PDE is simply a Monge-Ampère PDE whose coefficients are independent of z. We also note that it is possible for two symplectic Monge-Ampère PDEs to be contact equivalent but not symplectically equivalent, since the group of contact transformations is larger than the group of symplectomorphisms.

While second-order PDEs are represented by effective 2-forms on M (or $\widetilde{M}$), first-order PDEs are represented by functions. Specifically, a nondegenerate function $H : \widetilde{M} \to \mathbb{R}$ determines a 1-parameter family of first-order PDEs of the form

$$H(x, y, f, f_x, f_y) = c,$$

one for each constant $c \in \mathbb{R}$. Similarly, a nondegenerate function $h : M \to \mathbb{R}$ determines a 1-parameter family of first-order PDEs of the form

$$h(x, y, f_x, f_y) = c.$$

Note that for any partial prolongation $\rho : \widetilde{M} \to M$, the function $H = \rho^*(h)$ determines a 1-parameter family of first-order PDEs on $(\widetilde{M}, \tilde{\theta})$ in an obvious way.

3. Intermediate differential equations

We can associate to any Monge-Ampère PDE $(\widetilde{M}, \tilde{\theta}, \widetilde{\Omega})$ a collection of first-order PDEs known as *intermediate differential equations*, defined as follows.

Definition 3.1. An *intermediate differential equation* (IDE) for a Monge-Ampère PDE $(\widetilde{M}, \tilde{\theta}, \widetilde{\Omega})$ is a nondegenerate function $H : \widetilde{M} \to \mathbb{R}$ such that solutions for the family of first-order PDEs determined by H are also solutions for $(\widetilde{M}, \tilde{\theta}, \widetilde{\Omega})$. In terms of differential ideals, this condition may be expressed as

$$(\tilde{\theta}, d\tilde{\theta}, \widetilde{\Omega}) \subset (\tilde{\theta}, d\tilde{\theta}, dH). \tag{3.1}$$

IDEs are more commonly called *intermediate integrals.* This notion is originally due to Goursat [Gou96] and has been widely studied; see, e.g., [Cle00], [LR90], [LRC93], [Zil99].

Classically, an IDE for a *symplectic* Monge-Ampère PDE (M, ω, Ω) is defined to be an IDE for any partial prolongation. But because the partial prolongation construction is non-canonical, it would be preferable to define IDEs for symplectic Monge-Ampère PDEs as objects on M, independent of any particular partial prolongation. In the remainder of this section, we will show how this may be accomplished. After that, we proceed with our main objective: to show that the set of IDEs can be used to classify certain symplectic Monge-Ampère PDEs up to symplectic equivalence.

Relative to a partial prolongation $\rho : \widetilde{M} \to M$, IDEs can be distinguished as *cylindrical intermediate equations (CIEs)* or *graph-like intermediate equations (GIEs)*, depending upon their relationship with the $\mathbb{R}$-fibers of ρ.

Provisional Definition 3.2. Given a symplectic Monge-Ampère PDE (M, ω, Ω), a partial prolongation $\rho : \widetilde{M} \to M$, and an IDE $H : \widetilde{M} \to \mathbb{R}$, we say that H is

- *cylindrical* if the level sets $H = c$ are ruled by the $\mathbb{R}$-fibers of ρ (i.e., if $H = \rho^*(h)$ for some function $h : M \to \mathbb{R}$);
- *graph-like* if the level sets $H = c$ are transverse to the $\mathbb{R}$-fibers of ρ.

Remark 3.3. The term "graph-like" is motivated by the fact that, by the implicit function theorem, any first-order PDE $H = c$ in the family determined by a graph-like IDE $H : \widetilde{M} \to \mathbb{R}$ can locally be written in the form $z = \rho^*(h_c)$ for some function $h_c : M \to \mathbb{R}$.

Now we will show how IDEs for a symplectic Monge-Ampère PDE (M, ω, Ω) may be realized as objects on M. This discussion will make use of the observation that condition (3.1) is equivalent to

$$\widetilde{\Omega} \in (\tilde{\theta}, d\tilde{\theta}, dH). \tag{3.2}$$

First, consider a CIE $H = \rho^*(h)$, $h : M \to \mathbb{R}$. Condition (3.2) may be written in the form

$$\rho^*(\Omega) \in (\tilde{\theta}, \rho^*(\omega), \rho^*(dh)). \tag{3.3}$$

Since $\Lambda^2(\widetilde{M})$ admits the direct-sum decomposition

$$\Lambda^2(\widetilde{M}) = \big((\tilde{\theta}) \otimes \rho^*(T^*M)\big) \oplus \big(\rho^*(\Lambda^2(T^*M))\big),$$

(3.3) holds if and only if

$$\rho^*(\Omega) \in \big(\rho^*(\omega), \rho^*(dh)\big),$$

i.e., if and only if $\Omega \in (\omega, dh)$. In terms of ideals, this condition may be written as

$$(\omega, \Omega) \subset (\omega, dh). \tag{3.4}$$

Note that this condition is independent of the choice of partial prolongation $\rho : \widetilde{M} \to M$.

Locally, a symplectic Monge-Ampère PDE and its partial prolongation have identical coordinate expressions. Moreover, the condition that $H : \widetilde{M} \to \mathbb{R}$ is a CIE is simply that H is independent of z; i.e., $H = h(x, y, p, q)$. Conditions (3.1) and (3.4) both have the interpretation that any solution of a first-order PDE $h = c$ is also a solution of the symplectic Monge-Ampère PDE (2.2).

Next, consider the GIEs; we begin by proving a useful lemma.

Lemma 3.4. *The collection of GIEs for a symplectic Monge-Ampère PDE is generated by functions of the form $H = z - h(x, y, p, q)$.*

Proof. Let $H : \widetilde{M} \to \mathbb{R}$ be a GIE, and fix an arbitrary constant $c_0 \in \mathbb{R}$ in the range of H. By the implicit function theorem, the first-order PDE $H = c_0$ can locally be written in the form

$$z = h(x, y, p, q). \tag{3.5}$$

(Note that a different choice $\hat{c}_0$ for the constant will, in general, yield a different function $\hat{h}$ that does not necessarily differ from h by an additive constant.)

Let $z = f(x, y)$ be any solution for (3.5). Since H is an IDE, $z = f(x, y)$ is also a solution for the symplectic Monge-Ampère PDE (2.3). Now, for any $c \in \mathbb{R}$, the function $\hat{z} = f(x, y) + c$ is a solution for the first-order PDE

$$z = h(x, y, p, q) + c;$$

moreover, because (2.2) is symplectic (i.e., its coefficients are independent of the variable z), $\hat{z}$ is also a solution of (2.2). Therefore, all solutions for the 1-parameter family of first-order PDEs

$$z - h(x, y, p, q) = c$$

are also solutions for (2.2), and hence the function

$$\widehat{H} = z - h(x, y, p, q)$$

is an IDE.

In summary, we have shown that any first order PDE $H = c_0$ in the 1-parameter family determined by a GIE H can be rewritten as a first-order PDE in the 1-parameter family determined by a GIE of the form $\widehat{H} = z - h(x, y, p, q)$. The statement of the lemma follows immediately. $\square$

As a consequence of Lemma 3.4, it suffices to consider GIEs of the form $H = z - \rho^*(h)$, $h : M \to \mathbb{R}$. Condition (3.2) may be written in the form

$$\begin{aligned} \rho^*(\Omega) &\in (\tilde{\theta}, \rho^*(\omega), dz - \rho^*(dh)) \\ &= (\tilde{\theta}, \rho^*(\omega), \rho^*(\theta - dh)). \end{aligned} \tag{3.6}$$

(Recall that, in local symplectic coordinates, $\theta = p\,dx + q\,dy$.) By the same argument as in the CIE case, (3.6) holds if and only if

$$\rho^*(\Omega) \in (\rho^*(\omega), \rho^*(\theta - dh)),$$

i.e., if and only if $\Omega \in (\omega, \theta - dh)$. In terms of ideals, this condition may be written as

$$(\omega, \Omega) \subset (\omega, \theta - dh). \tag{3.7}$$

Note that this condition is independent of the choice of partial prolongation $\rho : \widetilde{M} \to M$. We will denote the 1-form $\theta - dh$ by θ_h.

For both CIEs and GIEs, the important object is not the function h, but rather a 1-form: dh in the case of a CIE or θ_h in the case of a GIE. These 1-forms may locally be distinguished by the conditions that

- $d(dh) = 0$ – i.e., dh is exact;

- $d(\theta_h) = \omega$.

We are now ready to define CIEs and GIEs as objects on M:

Definition 3.5. Let (M, ω, Ω) be a symplectic Monge-Ampère PDE. An *intermediate differential equation (IDE)* for (M, ω, Ω) is a 1-form α on M satisfying the conditions that:

- $d\alpha = \lambda\omega, \qquad \lambda \in \{0, 1\}$,
- $(\omega, \Omega) \subset (\omega, \alpha)$.

If $\lambda = 0$, we say that α is a *cylindrical intermediate equation (CIE)*; if $\lambda = 1$, we say that α is a *graph-like intermediate equation (GIE).*

4. The symplectic characteristic variety

We now introduce a fundamental invariant for symplectic Monge-Ampère PDEs, called the *symplectic characteristic variety.* This object was originally introduced by Lychagin; see, e.g., [Lyc85]. It will play an important role in our classification process.

Definition 4.1. For $e \in M$, the *symplectic characteristic variety* SCV_e of (M, ω, Ω) at e is the cone

$$\{\mathbf{v} \in T_eM \mid (\mathbf{v} \lrcorner \omega) \wedge (\mathbf{v} \lrcorner \Omega) = 0\}. \tag{4.1}$$

Notice that if one takes any $\mathbf{v} \in T_eM - \mathrm{SCV}_e$, then there exists a unique integral element $E_{\mathbf{v}} \subset T_eM$, necessarily containing $\mathbf{v}$, on which both 1-forms $(\mathbf{v} \lrcorner \omega)$ and $(\mathbf{v} \lrcorner \Omega)$ vanish. Consequently, SCV_e can be interpreted as the set of directions in T_eM for which the extension to an integral element is *not* unique. We note that the classical characteristic directions for a solution $s : D \to M$ are given by the intersection of the 2-planes tangent to the image of s with SCV_e [Kos91].

We will generally find it more convenient to work with the "symplectically dual" object

$$\mathrm{SCV}^*_e = \{\mathbf{v} \lrcorner \omega \mid \mathbf{v} \in \mathrm{SCV}_e\}.$$

We will use the term "symplectic characteristic variety" to refer to either SCV_e or SCV^*_e; it should be clear from the context which object is meant.

Proposition 4.2. *Let α be a nowhere-zero 1-form on M satisfying*

$$d\alpha = \lambda\,\omega,\ \ \lambda \in \{0, 1\}.$$

Then α is an IDE for the symplectic Monge-Ampère PDE (M, ω, Ω) if and only if $\alpha \in \mathrm{SCV}^*$.

Proof. Let X_α be the unique vector field on M satisfying

$$X_\alpha \lrcorner \omega = -\alpha.$$

Note that $\alpha \in \mathrm{SCV}^*$ if and only if $X_\alpha \in \mathrm{SCV}$.

Suppose that α is an IDE for (M, ω, Ω). Observe that the system (ω, α) is differentially closed: $d\alpha = \lambda\omega$, where λ is either 0 or 1, and $d\omega = 0$.

Furthermore, X_α is a Cauchy characteristic vector field for this system; i.e., $X_\alpha \lrcorner (\omega, \alpha) \subset (\omega, \alpha)$.

By definition, $(\omega, \Omega) \subset (\omega, \alpha)$; therefore, $X_\alpha \lrcorner \Omega \in (\omega, \alpha)$. Since $X_\alpha \lrcorner \Omega$ is a 1-form, it follows that it must be a multiple of $\alpha = -X_\alpha \lrcorner \omega$. Therefore,

$$(X_\alpha \lrcorner \omega) \wedge (X_\alpha \lrcorner \Omega) = 0,$$

and $X_\alpha \in \mathrm{SCV}$, as desired.

Conversely, suppose that $\alpha \in \mathrm{SCV}^*$. Then $X_\alpha \in \mathrm{SCV}$, and so

$$(X_\alpha \lrcorner \omega) \wedge (X_\alpha \lrcorner \Omega) = 0.$$

Therefore,

$$(X_\alpha \lrcorner \Omega) = \mu (X_\alpha \lrcorner \omega) = -\mu\, \alpha$$

for some function μ on M. Now consider the 2-form $\Omega + \mu\, \omega$, and compute its wedge product with α (recalling that $\Omega \wedge \omega = 0$):

$$\begin{aligned} \alpha \wedge (\Omega + \mu\, \omega) &= -(X_\alpha \lrcorner \omega) \wedge (\Omega + \mu\, \omega) \\ &= -X_\alpha \lrcorner [\omega \wedge (\Omega + \mu\, \omega)] + \omega \wedge [X_\alpha \lrcorner (\Omega + \mu\, \omega)] \\ &= -X_\alpha \lrcorner (\mu\, \omega \wedge \omega) - \omega \wedge (2\mu\, \alpha) \\ &= \mu(\alpha \wedge \omega + \omega \wedge \alpha) - 2\mu\, \omega \wedge \alpha \\ &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Omega + \mu\, \omega \equiv 0 \pmod{\alpha} \\ \Rightarrow\ \ &\Omega \equiv 0 \pmod{\omega,\, \alpha}. \end{aligned}$$

It follows that

$$(\omega, \Omega) \subset (\omega, \alpha),$$

and α is an IDE for (M, ω, Ω), as desired. □

5. Elliptic, hyperbolic, and parabolic types

We now describe a well-known local invariant of a symplectic Monge-Ampère PDE (M, ω, Ω). (See, e.g., [LRC93], [IL03].) Since every 4-form on M is a multiple of $\omega \wedge \omega$, we can write $\Omega \wedge \Omega = \tau\, \omega \wedge \omega$ for some real-valued function $\tau : M \to \mathbb{R}$. (Note that τ is well-defined only up to multiplication by a positive function.) In terms of the coordinate representation (2.2), we may write $\tau(x, y, p, q) = AC - B^2 - DE$. A point $e \in M$ is called:

- an *elliptic point* if $\tau(e) > 0$;
- a *hyperbolic point* if $\tau(e) < 0$;
- a *parabolic point* if $\tau(e) = 0$ and $\Omega_e \neq 0$;
- a *zero point* if $\Omega_e = 0$;
- a *parabolic point of type change* if e is a parabolic point and τ is not identically zero on any neighborhood of e;
- *a zero point of type change* if $\Omega_e = 0$ and τ is not identically zero on any neighborhood of e.

A *local $\mathcal{C}^\infty$-coframing* on an open subset $U \subset M$ is a set of four 1-forms $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ that comprises a basis for $T^*_e M$ at each point $e \in U$. We will say that such a coframing is *symplectic* if

$$\omega = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4. \tag{5.1}$$

The following observations are well-known; see [IL03] for an exposition. (A similar normal form appears in [Kus98].)

(a) If e is an elliptic point, then $\mathrm{SCV}_e = \{0\}$, and there exists a local $\mathcal{C}^\infty$-symplectic coframing such that $\Omega_e = \omega_2 \wedge \omega_3 + \omega_1 \wedge \omega_4$.

(b) If e is a hyperbolic point, then SCV_e consists of a pair of 2-planes V_e, $\overline{V}_e$ satisfying $V_e \cap \overline{V}_e = \{0\}$ and $\omega(V_e, \overline{V}_e) = 0$. Furthermore, there exists a local $\mathcal{C}^\infty$-symplectic coframing such that $\Omega_e = \omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_4$. In terms of this coframing, we have $V_e = \{\omega_1, \omega_2\}^\perp$, $\overline{V}_e = \{\omega_3, \omega_4\}^\perp$. Every integral element in $T_e M$ intersects SCV_e in two lines, one lying in V_e and the other in $\overline{V}_e$. In this case,

$$\mathrm{SCV}^*_e = V_e^\perp \cup \overline{V}_e^\perp = \{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4\}.$$

(c) If e is a parabolic point, then SCV_e consists of a single 2-plane on which ω vanishes. Furthermore, there exists a local $\mathcal{C}^\infty$-symplectic coframing such that $\Omega_e = \omega_2 \wedge \omega_3$. In terms of this coframing, we have $\mathrm{SCV}_e = \{\omega_2, \omega_3\}^\perp$. Every integral element in $T_e M$ either intersects SCV_e in a line or coincides with SCV_e. In this case,

$$\mathrm{SCV}^*_e = V_e^\perp = \{\omega_2, \omega_3\}.$$

(d) If e is a zero point, then $\mathrm{SCV}_e = T_e M$.

6. Normal forms for hyperbolic PDEs with intermediate equations

Much study has been given to the problem of classifying certain types of PDEs up to contact equivalence. Results along these lines regarding intermediate equations include [LRC93], in which the authors show that any PDE posessing intermediate integrals of a certain type must be contact equivalent to either the heat equation or the flat wave equation; and [Zil99], in which Zil'bergleit classifies intermediate integrals for Monge-Ampère PDEs of constant type under the additional restriction that the PDE can be described by an effective form Ω which is closed. (This is a significant restriction; for instance, it implies that the PDE has a nontrivial conservation law of order 1.)

In this section, we construct models for hyperbolic symplectic Monge-Ampère PDEs with CIEs. These results should be considered preliminary to those of §7, in which we will construct similar models for type-changing PDEs. Our normal forms will be evolution equations of the form

$$\Omega = B\,(dq \wedge dt - dp \wedge dx) + C\,dx \wedge dq \tag{6.1}$$

for $\mathcal{C}^\infty$-functions B, C. The corresponding coordinate representation is

$$2B(x,t,f_x,f_t)\,f_{xt} + C(x,t,f_x,f_t)\,f_{tt} = 0.$$

Note that, in the tradition of evolution equations, we are using the variable t in place of the variable y that appears in (2.1).

Since (M,ω,Ω) is hyperbolic, there exists a local symplectic coframing $\{\omega_1,\omega_2,\omega_3,\omega_4\}$ on M such that

$$\omega = \omega_1\wedge\omega_2 + \omega_3\wedge\omega_4 \tag{6.2}$$
$$\mathrm{SCV}^* = V^\perp \cup \overline{V}^\perp = \{\omega_1,\omega_2\}\cup\{\omega_3,\omega_4\}.$$

Recall that any CIE is a closed 1-form $\alpha = dh$ on M. It follows from Proposition 4.2 that (M,ω,Ω) has a CIE if and only if one of the components $V^\perp$, $\overline{V}^\perp$ of the symplectic characteristic variety contains a nonzero integrable subsystem.

The following theorem provides a preliminary classification of hyperbolic symplectic Monge-Ampère PDEs for which each component of the characteristic system contains a CIE.

Theorem 6.1. *If (M,ω,Ω) is hyperbolic and $V^\perp$, $\overline{V}^\perp$ each contain a 1-dimensional integrable subsystem, then at any point $e\in M$, (M,ω,Ω) is locally symplectically equivalent to (6.1), where the point e corresponds to $(x,t,p,q) = (0,0,1,1)$ and $B\neq 0$. This equation has CIEs of the form $d(h(q))$, $d(\bar{h}(x))$, where h, $\bar{h}$ are arbitrary nondegenerate $\mathcal{C}^\infty$-functions of one variable.*

Remark 6.2. The choice of local coordinates satisfying $(x,t,p,q) = (0,0,1,1)$ at e is in keeping with with the local normal forms that we will construct near parabolic and zero points in Section 7.

Proof. We can choose a local coframing of the form (6.2) so that ω_1 spans an integrable subsystem of $V^\perp$ and ω_3 spans an integrable subsystem of $\overline{V}^\perp$. Multiplying by nonvanishing functions, we can arrange that $\omega_1 = dq$ and $\omega_3 = dx$ for some independent functions x and q. Furthermore,

$$dx\wedge dq\wedge\omega = 0.$$

It follows from a theorem of Liouville [Bry95, Theorem 3] that we can complete (x,q) to a local symplectic coordinate system (x,t,p,q), so that

$$\omega = \omega_1\wedge\omega_2 + \omega_3\wedge\omega_4 = dq\wedge dt + dp\wedge dx. \tag{6.3}$$

Now $\omega\wedge dx = dq\wedge dt\wedge dx = dq\wedge\omega_2\wedge dx$; therefore,

$$\omega_2 \equiv dt + \gamma\,dx \pmod{dq}.$$

Similarly, $\omega\wedge dq = dp\wedge dx\wedge dq = dx\wedge\omega_4\wedge dq$; therefore,

$$\omega_4 \equiv -dp + \delta\,dq \pmod{dx}.$$

Next, equation (6.3) implies that $\delta = \gamma$. If we set $Q = \frac{C}{2B} = \delta = \gamma$ (with $B \neq 0$), then since Ω is only determined up to a nonzero multiple, we can write

$$\begin{aligned} \omega &= dq \wedge (dt - Q\,dx) + (dp + Q\,dq) \wedge dx \\ \Omega &= B\Big(dq \wedge (dt - Q\,dx) - (dp + Q\,dq) \wedge dx\Big) \\ &= B(dq \wedge dt - dp \wedge dx) + C\,dx \wedge dq, \end{aligned} \tag{6.4}$$

as claimed. Observe that, for any nondegenerate $\mathcal{C}^\infty$-functions $h, \bar{h}$ of one variable, we have

$$d(h(q)) = h'(q)\,dq \in V^\perp, \qquad d(\bar{h}(x)) = \bar{h}'(x)\,dx \in \overline{V}^\perp;$$

therefore, by Proposition 4.2, $d(h(q))$ and $d(\bar{h}(x))$ are CIEs for (6.1). □

Note that in the course of this proof, we demonstrated the following corollary.

Corollary 6.3. *Under the conditions of Theorem 6.1, the components of* SCV* *take the form*

$$V^\perp = \{dq,\, dt - Q\,dx\}, \qquad \overline{V}^\perp = \{dx,\, dp + Q\,dq\}.$$

For the remainder of the paper, we will consider refinements of the normal form (6.1) under several additional assumptions. It will be useful to observe that for this class of PDEs, the Frobenius system $\{\omega_1, \omega_3\} = \{dx, dq\}$ defines a local fibration $\pi : M \to \mathbb{R}^2$ given by

$$\pi(x, t, p, q) = (x, q).$$

This fibration gives M the structure of the cotangent bundle $T^*\mathbb{R}^2$ with ω as its canonical symplectic form. Moreover, we have the product decomposition

$$T^*\mathbb{R}^2 = T^*\mathbb{R} \times T^*\mathbb{R},$$

with $\pi : T^*\mathbb{R} \times T^*\mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by

$$\pi\big[(x, p) \times (q, -t)\big] = (x) \times (q).$$

In the cases of Theorems 6.4 and 7.5, this fibration is, in fact, canonical.

The subgroup of symplectomorphisms which preserve this product structure is generated by:

- *Product diffeomorphisms* $(x, t, p, q) \mapsto (\chi(x), t/\mathcal{Q}'(q), p/\chi'(x), \mathcal{Q}(q))$, where $\chi(x)$, $\mathcal{Q}(q)$ are $\mathcal{C}^\infty$-diffeomorphisms of $\mathbb{R}$. Under such a symplectomorphism, the function Q in (6.4) transforms by $Q \mapsto \mathcal{Q}'(q)\chi'(x)Q$.
- *F-translations* $(x, t, p, q) \mapsto (x, t - F_q, p + F_x, q)$, where $F(x, q)$ is any $\mathcal{C}^\infty$-function. Under such a symplectomorphism, the function Q transforms by $Q \mapsto Q(x, t - F_q, p + F_x, q) + F_{xq}$.

The following theorem indicates that if each component of SCV* contains both a CIE and a GIE, then there is a refined normal form for the PDE (6.1) depending only on one real parameter a.

Theorem 6.4. *Let (M, ω, Ω) be as in Theorem 6.1, and suppose that $V^\perp$, $\overline{V}^\perp$ each contain at least one GIE. Then at any point $e \in M$, (M, ω, Ω) is locally symplectically equivalent to* (6.1)*, where e corresponds to* $(0, 0, 1, 1)$ *and $Q_{pt} = a$ for an invariant real constant a. Specifically:*

(1) If $a \neq 0$, then (M, ω, Ω) has local normal form (6.1) *with*

$$Q = \frac{C}{2B} = ap(t + 1).$$

The GIEs belonging to $V^\perp$, $\overline{V}^\perp$ are represented by the 1-forms $\theta_h, \theta_{\bar{h}}$, where

$$\begin{aligned} h &= t\,q + \frac{1}{a}\ln(t + 1) + k(q) \\ \bar{h} &= (t + 1)\,q + \frac{1}{a}\ln|p| + \bar{k}(x), \end{aligned}$$

and k and $\bar{k}$ are arbitrary $\mathcal{C}^\infty$-functions of one variable.

(2) If $a = 0$, then (M, ω, Ω) has local normal form (6.1) *with*

$$Q = \frac{C}{2B} = \frac{p - t}{x + q + 1}.$$

The GIEs belonging to $V^\perp$, $\overline{V}^\perp$ are represented by the 1-forms $\theta_h, \theta_{\bar{h}}$, where

$$\begin{aligned} h &= t\,q + t(x + q + 1) + k(q) \\ \bar{h} &= t\,q - p(x + q + 1) + \bar{k}(x), \end{aligned}$$

and k and $\bar{k}$ are arbitrary $\mathcal{C}^\infty$-functions of one variable.

Proof. Choose a local coframing as in the proof of Theorem 6.1. By Corollary 6.3, we can write the GIEs belonging to $V^\perp$ and $\overline{V}^\perp$ as

$$\begin{aligned} \theta_h &= \alpha\,dq + \beta\,(dt - Q\,dx) \\ \theta_{\bar{h}} &= \bar{\alpha}\,dx + \bar{\beta}\,(dp + Q\,dq) \end{aligned}$$

for some $\mathcal{C}^\infty$-functions α, β, $\bar{\alpha}$, $\bar{\beta}$. Thus we have

$$d\big[\alpha\,dq + \beta\,(dt - Q\,dx)\big] = d\big[\bar{\alpha}\,dx + \bar{\beta}\,(dp + Q\,dq)\big] = \omega. \tag{6.5}$$

Since $d(t\,dq - p\,dx) = -\omega = -dp \wedge dx - dq \wedge dt$, this is equivalent to

$$\begin{aligned} (\alpha + t)\,dq + \beta\,dt - (\beta\,Q + p)\,dx &= dg(x, t, q) \\ (\bar{\alpha} - p)\,dx + \bar{\beta}\,dp + (\bar{\beta}\,Q + t)\,dq &= d\bar{g}(x, p, q) \end{aligned} \tag{6.6}$$

for some $\mathcal{C}^\infty$-functions $g, \bar{g}$. Six of the twelve integrability conditions for (6.6) are:

$$\beta_p = 0 \qquad \bar{\beta}_t = 0 \tag{6.7}$$

$$(\beta\,Q + p)_p = 0 \qquad (\bar{\beta}\,Q + t)_t = 0 \tag{6.8}$$

$$\beta_x + (\beta\,Q + p)_t = 0 \qquad \bar{\beta}_q - (\bar{\beta}\,Q + t)_p = 0. \tag{6.9}$$

As immediate consequences of equations (6.7) and (6.8), we have:

$$Q_p = -\frac{1}{\beta(x,t,q)}, \qquad Q_t = -\frac{1}{\bar{\beta}(x,p,q)}, \tag{6.10}$$

which imply that

$$Q_{pp} = 0 = Q_{tt}. \tag{6.11}$$

Substituting $\beta = -\frac{1}{Q_p}$ and $\bar{\beta} = -\frac{1}{Q_t}$ into equations (6.9), we find that

$$Q\,Q_{pt} - Q_p\,Q_t + Q_{px} = 0 \tag{6.12}$$

$$Q\,Q_{pt} - Q_p\,Q_t - Q_{tq} = 0. \tag{6.13}$$

Differentiating (6.12) with respect to t shows that $Q_{ptx} = 0$, and differentiating (6.13) with respect to p shows that $Q_{ptq} = 0$. Additionally, equation (6.11) implies that $Q_{ptp} = Q_{ptt} = 0$; therefore, we conclude that Q_{pt} is constant. Hence there is a real number a such that

$$Q_{pt} = a = -\left(\frac{1}{\beta}\right)_t = -\left(\frac{1}{\bar{\beta}}\right)_p. \tag{6.14}$$

Equations (6.11) and (6.14) imply that there exist functions $b(x,q)$, $c(x,q)$, and $\tilde{Q}(x,q)$ for which

$$Q(x,t,p,q) = apt - b(x,q)p - c(x,q)t + \tilde{Q}(x,q). \tag{6.15}$$

Hence,

$$\frac{1}{\beta} = -at + b(x,q), \qquad \frac{1}{\bar{\beta}} = -ap + c(x,q).$$

The functions $b(x,q)$, $c(x,q)$, and $\tilde{Q}(x,q)$ are not arbitrary. When we substitute equation (6.15) into (6.12) and (6.13), we find that

$$\begin{aligned} b_x(x,q) + c(x,q)\,b(x,q) &= a\tilde{Q}(x,q) \\ c_q(x,q) - c(x,q)\,b(x,q) &= -a\tilde{Q}(x,q). \end{aligned} \tag{6.16}$$

Adding these equations, we obtain $b_x + c_q = 0$, which implies that (locally) there exists a function $u(x,q)$ satisfying

$$b(x,q) = u_q(x,q), \qquad c(x,q) = -u_x(x,q), \tag{6.17}$$

$$u_{xq} - u_x u_q = a\tilde{Q}(x,q). \tag{6.18}$$

We need to consider separately the cases $a \neq 0$ and $a = 0$.

First, suppose that $a \neq 0$. Then

$$\begin{aligned} Q(x,t,p,q) &= apt - u_q(x,q)\,p + u_x(x,q)\,t + \frac{1}{a}\big[u_{xq}(x,q) - u_x(x,q)\,u_q(x,q)\big] \\ &= a\left(p + \frac{u_x}{a}\right)\left(t - \frac{u_q}{a}\right) + \frac{u_{xq}}{a}. \end{aligned}$$

From this last expression, it is clear that applying an F-translation with $F(x,q) = -\frac{u(x,q)}{a} - q$ will transform Q to

$$Q(x,t,p,q) = ap(t+1). \tag{6.19}$$

From this normal form, a few calculations using equations (6.6) give

$$\begin{aligned} g(t,q) &= -\frac{1}{a}\,\ln(t+1) - k(q) \\ \bar{g}(x,p) &= -\frac{1}{a}\,\ln|p| - q - \bar{k}(x), \end{aligned} \tag{6.20}$$

where k and $\bar{k}$ are arbitrary $\mathcal{C}^\infty$-functions of one variable.

Next, suppose that $a = 0$. Equations (6.15), (6.17) and (6.18) imply that

$$Q(x,t,q,p) = -u_q(x,q)\,p + u_x(x,q)\,t + \widetilde{Q}(x,q), \tag{6.21}$$

where $u(x,q)$ satisfies

$$u_{xq}(x,q) = u_x(x,q)\,u_q(x,q). \tag{6.22}$$

By applying an F-translation, we can transform $\widetilde{Q}$ to zero.

The general solution of (6.22) is

$$u(x,q) = -\ln|\phi(q) + \psi(x)|;$$

moreover, the condition that $V^\perp$ and $\overline{V}^\perp$ each contains a GIE implies that neither $V^\perp$ nor $\overline{V}^\perp$ is completely integrable. Therefore, $u_q = -Q_p \neq 0$ and $u_x = Q_t \neq 0$; it follows that $\phi'(q) \neq 0$, $\psi'(x) \neq 0$.

Under a product diffeomorphism $(\chi(x), \mathcal{Q}(q))$, the function

$$Q(x,t,p,q) = \frac{\phi'(q)}{(\phi(q)+\psi(x))}\,p - \frac{\psi'(x)}{(\phi(q)+\psi(x))}\,t$$

is transformed to

$$Q(x,t,p,q) = \frac{\phi'(\mathcal{Q}(q))\mathcal{Q}'(q)}{(\phi(\mathcal{Q}(q))+\psi(\chi(x)))}\,p - \frac{\psi'(\chi(x))\chi'(x)}{(\phi(\mathcal{Q}(q))+\psi(\chi(x)))}\,t.$$

Since $\phi'(q) \neq 0$, $\psi'(x) \neq 0$, we can choose $\chi, \mathcal{Q}$ so that

$$\phi(\mathcal{Q}(q)) = q, \qquad \psi(\chi(x)) = x+1.$$

Applying this product diffeomorphism transforms Q to

$$Q(x,t,p,q) = \frac{p-t}{x+q+1}.$$

From this normal form, a few calculations using equations (6.6) give

$$\begin{aligned} g(x,t,q) &= -t(x+q+1) - k(q) \\ \bar{g}(x,p,q) &= p(x+q+1) - \bar{k}(x), \end{aligned} \tag{6.23}$$

where k and $\bar{k}$ are arbitrary $\mathcal{C}^\infty$-functions of one variable.

Observe that from equations (6.5) and (6.6), the GIEs are given by

$$\begin{aligned} \alpha\,dq + \beta\,(dt - Q\,dx) &= p\,dx + q\,dt - d(tq - g) = \theta_{tq-g} \\ \bar{\alpha}\,dx + \bar{\beta}\,(dp + Q\,dq) &= p\,dx + q\,dt - d(tq - \bar{g}) = \theta_{tq-\bar{g}}. \end{aligned}$$

When $a \neq 0$, equations (6.20) yield

$$h = tq - g = t\,q + \frac{1}{a}\ln(t+1) + k(q)$$
$$\bar{h} = tq - \bar{g} = (t+1)\,q + \frac{1}{a}\ln|p| + \bar{k}(x).$$

When $a = 0$, equations (6.23) yield

$$h = tq - g = t\,q + t(x+q+1) + k(q)$$
$$\bar{h} = tq - \bar{g} = t\,q - p(x+q+1) + \bar{k}(x).$$

This completes the proof. □

Next, we consider the asymmetric case where $V^\perp$ contains both a CIE and a GIE, while $\overline{V}^\perp$ is completely integrable.

Theorem 6.5. *Let (M, ω, Ω) be as in Theorem 6.1, and suppose that $V^\perp$ contains at least one GIE and $\overline{V}^\perp$ is completely integrable. Then at any point $e \in M$, (M, ω, Ω) is locally symplectically equivalent to* (6.1)*, where e corresponds to $(0,0,1,1)$ and $Q = \frac{C}{2B} = \frac{p}{2}$. The CIEs belonging to $\overline{V}^\perp$ are represented by the 1-form $d\bar{h}$, where*

$$\bar{h} = \bar{h}(x, p^2 e^q),$$

and $\bar{h}$ is an arbitrary $\mathcal{C}^\infty$-function of two variables. The GIEs belonging to $V^\perp$ are represented by the 1-form θ_h, where

$$h = t(q+2) + k(q),$$

and k is an arbitrary $\mathcal{C}^\infty$-function of one variable.

Proof. This proof will be very similar to that of Theorem 6.4. Choose a local coframing as in the proof of Theorem 6.1. Then

$$\overline{V}^\perp = \{\omega_3, \omega_4\} = \{dx, dp + Q\,dq\}.$$

The complete integrability of $\overline{V}^\perp$ implies that $Q_t = 0$.

By Corollary 6.3, we can write the GIE belonging to $V^\perp$ as

$$\theta_h = \alpha\,dq + \beta\,(dt - Q\,dx)$$

for some $\mathcal{C}^\infty$-functions α, β with $\beta \neq 0$. Thus we have

$$d\big[\alpha\,dq + \beta\,(dt - Q\,dx)\big] = \omega. \tag{6.24}$$

Since $d(t\,dq - p\,dx) = -\omega = -dp \wedge dx - dq \wedge dt$, this is equivalent to

$$(\alpha + t)\,dq + \beta\,dt - (\beta\,Q + p)\,dx = dg(x,t,q) \tag{6.25}$$

for some $\mathcal{C}^\infty$-function g. Three of the six integrability conditions for (6.25) are:

$$\beta_p = 0 \tag{6.26}$$
$$(\beta\,Q + p)_p = 0 \tag{6.27}$$
$$\beta_x + (\beta\,Q + p)_t = 0. \tag{6.28}$$

As an immediate consequence of equations (6.26) and (6.27) and the fact that $Q_t = 0$, we have:

$$Q_p = -\frac{1}{\beta(x,q)}. \tag{6.29}$$

Substituting $\beta = -\frac{1}{Q_p}$ into equation (6.28), we find that

$$Q_{xp} = 0. \tag{6.30}$$

Equations (6.29) and (6.30) imply that there exists a function $\widetilde{Q}(x,q)$ for which

$$Q(x,t,p,q) = -\frac{p}{\beta(q)} + \widetilde{Q}(x,q). \tag{6.31}$$

Now apply a product diffeomorphism with $\chi(x) = x$ and $\mathcal{Q}(q)$ a solution to the ODE

$$\mathcal{Q}'(q) = -\tfrac{1}{2}\beta(\mathcal{Q}(q));$$

this transforms Q to the normal form

$$Q(x,t,p,q) = \frac{p}{2} + \widetilde{Q}(x,q).$$

(The transformed function $\widetilde{Q}(x,q)$ may differ from that in (6.31).)

Finally, let $F(x,q)$ be a solution for the linear hyperbolic PDE

$$F_{xq}(x,q) + \tfrac{1}{2}F_x(x,q) + \widetilde{Q}(x,q) = 0;$$

the corresponding F-translation will transform Q to

$$Q(x,t,q,p) = \frac{p}{2}.$$

From this normal form, a few calculations using equation (6.25) give

$$g(x,t,q) = -2t - k(q), \tag{6.32}$$

where k is an arbitrary $\mathcal{C}^\infty$-function of one variable.

Now consider the IDEs. We have

$$\overline{V}^\perp = \{dx, dp + \frac{p}{2}\,dq\} = \{dx, d(p^2e^q)\}.$$

Therefore, the CIEs belonging to $\overline{V}^\perp$ have the form $d\bar{h}$, where

$$\bar{h} = \bar{h}(x, p^2e^q),$$

and $\bar{h}$ is an arbitrary $\mathcal{C}^\infty$-function of two variables.

As for the GIEs, observe that from equations (6.24) and (6.25), the GIEs are are given by

$$\alpha\,dq + \beta\,(dt - Q\,dx) = p\,dx + q\,dt - d(t(q+2) + k(q)) = \theta_{t(q+2)+k(q)}.$$

This completes the proof. □

The following case is the classical prototype for the main results of this paper. It was known to Lie and Darboux and is included for completeness. Note that the normal form is equivalent to the classical wave equation $f_{xt} = 0$.

Theorem 6.6. *Let (M,ω,Ω) be as in Theorem 6.1, and suppose that $V^\perp$, $\overline{V}^\perp$ are both completely integrable. Then at any point $e \in M$, (M,ω,Ω) is locally symplectically equivalent to* (6.1)*, where e corresponds to $(0,0,1,1)$ and $Q = 1$. The CIEs are given by $dh, d\bar{h}$, where*

$$h = h(q, t-x), \qquad \bar{h} = \bar{h}(x, p+q), \tag{6.33}$$

and h, $\bar{h}$ are arbitrary $\mathcal{C}^\infty$-functions of 2-variables.

Proof. A similar argument to that given in the proof of Theorem 6.5 shows that

$$Q_p = Q_t = 0,$$

and so $Q = Q(x,q)$.

Let $F(x,q)$ be a solution for the linear hyperbolic PDE

$$F_{xq}(x,q) = Q(x,q) - 1.$$

The corresponding F-translation will transform Q to

$$Q(x,t,q,p) = 1.$$

Now consider the IDEs. We have

$$V^\perp = \{dq, dt - dx\}, \qquad \overline{V}^\perp = \{dx, dp + dq\}.$$

Therefore, the CIEs have the forms dh, $d\bar{h}$, where

$$h = h(q, t-x), \qquad \bar{h} = \bar{h}(x, p+q),$$

and h, $\bar{h}$ are arbitrary $\mathcal{C}^\infty$-functions of two variables. □

7. Main results: Normal forms for type-changing PDEs with intermediate equations

Here we will adapt the methodology of Section 6 to the case where the PDE (M,ω,Ω) has mixed type. Our key involutivity assumption will be that the type-changing locus is defined by a CIE. We note that some early results on normal forms for mixed type equations are given in [Kus92, Kus95]; however, these results are obtained under the assumption of a certain linear independence condition which is not satisfied for our equations.

In general, if $e \in M$ is a parabolic point of type change or a zero point, then SCV_e^* need not $\mathcal{C}^\infty$-extend to a neighborhood of e. Here we will use IDEs to locally characterize a class of evolution equations for which both components $V^\perp$, $\overline{V}^\perp$, which are defined on the hyperbolic locus, extend smoothly to the locus of parabolic points $\mathcal{P} \subset M$. For this class, $\mathcal{P}$ is a regular hypersurface, and parabolicity implies that $V^\perp$ coincides with $\overline{V}^\perp$ at each point of $\mathcal{P}$. This parabolic type-changing locus $\mathcal{P}$ will be *involutive* in the sense that every Lagrangian surface $s : D \to \mathcal{P} \subset M$ will also be a solution to (M,ω,Ω).

If the PDE also has a locus of zero points $\mathcal{Z} \subset M$, then $V^\perp$ and $\overline{V}^\perp$ will *not* $\mathcal{C}^\infty$-extend to the zero points; specifically, $\mathrm{SCV}^* = \{0\}$ at each point of $\mathcal{Z}$. Here we will assume that the zero locus will be involutive in the sense that $\mathcal{Z}$ is a $\mathcal{C}^\infty$ Lagrangian surface, and hence defines a degenerate solution for the PDE.

Definition 7.1. Let (M, ω, Ω) be a symplectic Monge-Ampère PDE for which every point of M is either a hyperbolic point, a parabolic point of type change, or a zero point. Suppose that the locus $\mathcal{P} \subset M$ of parabolic points is 3-dimensional, and that the locus $\mathcal{Z} \subset M$ of zero points is a 2-dimensional Lagrangian surface. We say that (M, ω, Ω) is *involutive type-changing of order* m, $m \in \mathbb{Z}_+$, if:

- the closure of $\mathcal{P}$ is equal to $\mathcal{P} \cup \mathcal{Z}$;
- there exists a $\mathcal{C}^\infty$-function $q : M \to \mathbb{R}$, with dq nowhere-vanishing, such that the zero locus of q is $\mathcal{P} \cup \mathcal{Z}$;
- the components $V^\perp, \overline{V}^\perp$ of SCV* each $\mathcal{C}^\infty$-extend (as rank 2 Pfaffian systems) to the parabolic locus $\mathcal{P}$;
- dq is a CIE belonging to $V^\perp$ at each point of $M - \mathcal{Z}$;
- $\Omega \wedge \Omega = \tau\, \omega \wedge \omega$, where $\tau = q^{2m}\hat{\tau}$, and $\hat{\tau}$ is $\mathcal{C}^\infty$ and nonvanishing.

We will say that (M, ω, Ω) is *involutive type-changing of order* m *with CIEs* if, in addition, there exists another function $x : M \to \mathbb{R}$ such that dx is a CIE belonging to $\overline{V}^\perp$ at each point of $M - \mathcal{Z}$, the pullback of dx to $\mathcal{Z}$ is a nonvanishing 1-form on $\mathcal{Z}$, and $dq \wedge dx$ is a nonvanishing 2-form on M.

We will now prove analogs of the theorems in Section 6 for involutive type-changing equations with CIEs. The following lemma will be useful.

Lemma 7.2. *Let* (M, ω, Ω) *be involutive type-changing of order* m *with CIEs.*

(1) In a neighborhood of any parabolic point $e \in M$*, we can complete* (x, q) *to a symplectic coordinate system* (x, t, p, q) *(i.e., a coordinate system for which* $\omega = dq \wedge dt + dp \wedge dx$*) such that* e *corresponds to the point* $(x, t, p, q) = (0, 0, 1, 0)$*.*

(2) In a neighborhood of any zero point $e \in M$*, we can complete* (x, q) *to a symplectic coordinate system* (x, t, p, q) *such that* e *corresponds to the point* $(x, t, p, q) = (0, 0, 0, 0)$*, and the zero locus near* e *has the form* $\mathcal{Z} = \{q = p = 0\}$*.*

Proof. Because dx and dq lie in distinct characteristic subsystems, the proof of (1) is immediate. For case (2), let (x, t, p, q) be any completion of (x, q) to local symplectic coordinates in a neighborhood of e such that e corresponds to $(0, 0, 0, 0)$. Since $\mathcal{Z}$ is a Lagrangian surface on which $q = 0$ and dx is nonvanishing, it must have the form

$$\mathcal{Z} = \{q = p - \varphi(x) = 0\}$$

for some function $\varphi(x)$ with $\varphi(0) = 0$. By making the symplectic coordinate transformation

$$(x, t, p, q) \mapsto (x, t, p + \varphi(x), q)$$

we can assume that $\mathcal{Z}$ has the form

$$\mathcal{Z} = \{q = p = 0\},$$

as desired. □

Theorem 7.3. *If* (M, ω, Ω) *is involutive type-changing of order* m *with CIEs, then at any point* $e \in \mathcal{P}$, (M, ω, Ω) *is locally symplectically equivalent to the normal form*

$$\Omega = q^m (dq \wedge dt - dp \wedge dx) + C\, dx \wedge dq, \tag{7.1}$$

where the point e *corresponds to* $(x, t, p, q) = (0, 0, 1, 0)$*, and* C *is a* $\mathcal{C}^\infty$*-function which is nonzero on the parabolic locus* $\mathcal{P}$*. The corresponding coordinate representation is*

$$2f_t^m f_{xt} + C(x, t, f_x, f_t) f_{tt} = 0.$$

Furthermore, at any point $e \in \mathcal{Z}$ *a similar conclusion holds, where the point* e *now corresponds to* $(x, t, p, q) = (0, 0, 0, 0)$*, with the additional condition that* C *is a* $\mathcal{C}^\infty$*-function which vanishes on the zero locus* $\mathcal{Z} = \{q = p = 0\}$*.*

Proof. Let dq, dx be the CIEs of Definition 7.1, and choose local symplectic coordinates (x, t, p, q) in a neighborhood of e as in Lemma 7.2. We can divide Ω by $\sqrt{|\hat{\tau}|}$ to arrange that $\tau = -q^{2m}$. Near this point, the characteristic systems are given by

$$V^\perp = \{dq,\ q^m\, dt - \tfrac{1}{2} C\, dx\}, \qquad \overline{V}^\perp = \{dx,\ q^m\, dp + \tfrac{1}{2} C\, dq\},$$

and we can write

$$\Omega = q^m (dq \wedge dt - dp \wedge dx) + C\, dx \wedge dq$$

for some $\mathcal{C}^\infty$ function C.

If $e \in \mathcal{P}$, then C must be nonzero on a neighborhood of e, or more precisely, on the intersection of the local symplectic coordinate neighborhood with $M - \mathcal{Z}$. If $e \in \mathcal{Z}$, then C must vanish precisely on the intersection of the local symplectic coordinate neighborhood with $\mathcal{Z}$. □

Note that we have the following analog of Corollary 6.3:

Corollary 7.4. *Under the conditions of Theorem 7.3, the components of* SCV* *take the form*

$$V^\perp = \{dq,\, q^m dt - \tfrac{1}{2} C\, dx\}, \qquad \overline{V}^\perp = \{dx,\, q^m dp + \tfrac{1}{2} C\, dq\}.$$

We will now prove analogs of Theorem 6.4, Theorem 6.5, and Theorem 6.6 for involutive type-changing PDEs with CIEs. For arbitrary order m these normal forms are rather unwieldy; therefore, we will restrict our attention to the case $m = 1$.

Theorem 7.5. *Let* (M, ω, Ω) *be involutive type-changing of order* 1 *with CIEs, and suppose that* $V^\perp$, $\overline{V}^\perp$ *each contain at least one GIE. Then at any point* e *in* $\mathcal{P}$ *or* $\mathcal{Z}$, (M, ω, Ω) *is locally symplectically equivalent to* (7.1)*, where* e *corresponds to* $(0, 0, 1, 0)$ *if* $e \in \mathcal{P}$ *and to* $(0, 0, 0, 0)$ *if* $e \in \mathcal{Z}$*, and* $C_{pt} = 2aq$ *for an invariant real constant* a*. Specifically:*

(1) If $a \neq 0$, then (M, ω, Ω) has local normal form (7.1) *with*

$$C = 2ap(tq + b)$$

for an invariant nonzero constant b. The GIEs belonging to $V^{\perp}$, $\overline{V}^{\perp}$ are represented by the 1-forms $\theta_h, \theta_{\bar{h}}$, where

$$h = t\,q + \frac{1}{a}\ln|tq + b| + k(q)$$

$$\bar{h} = tq + b\ln|q| + \frac{1}{a}\ln|p| + \bar{k}(x),$$

and k and $\bar{k}$ are arbitrary $\mathcal{C}^{\infty}$-functions of one variable.

(2) If $a = 0$, then $\mathcal{Z}$ is empty and (M, ω, Ω) has local normal form (7.1) *with*

$$C = \frac{2q\,(\phi'(q)p - t)}{\phi(q) + x + 1} + c_0(x)$$

for some nonvanishing $\mathcal{C}^{\infty}$ function $c_0(x)$ and some $\mathcal{C}^{\infty}$ function $\phi(q)$ satisfying $\phi(0) = 0$ and $\phi'(q) \neq 0$ when $q \neq 0$. The GIEs belonging to $V^{\perp}$, $\overline{V}^{\perp}$ are represented by the 1-forms $\theta_h, \theta_{\bar{h}}$, where

$$h = t\,q + \frac{1}{\phi'(q)}\left(t(\phi(q) + x + 1) - \frac{1}{2q}\int c_0(x)(\phi(q) + x + 1)\,dx\right) + k(q)$$

$$\bar{h} = t\,q - p(\phi(q) + x + 1) - \frac{1}{2}c_0(x)\left((x+1)\ln|q| + \int \frac{\phi(q)}{q}\,dq\right) + \bar{k}(x),$$

and k and $\bar{k}$ are arbitrary $\mathcal{C}^{\infty}$-functions of one variable.

Remark 7.6. Note that many of the GIEs above do not extend smoothly to the parabolic locus.

Proof. Choose local coordinates so that Ω has the normal form (7.1) with $m = 1$. An analogous argument to that given in the proof of Theorem 6.4 shows that

$$C(x, t, p, q) = 2q[apt - u_q(x, q)p + u_x(x, q)t + \tilde{Q}(x, q)]$$

for some functions $u(x, q)$, $\tilde{Q}(x, q)$ satisfying

$$u_{xq} - u_x u_q = a\tilde{Q}(x, q).$$

The crucial difference in this case is that the functions $u(x, q), \tilde{Q}(x, q)$ *cannot* both be continuous at $e \in \mathcal{P}$ or $e \in \mathcal{Z}$, because C cannot vanish identically on the set $\{q = 0\}$. This will limit our ability to normalize these functions via F-translations and product diffeomorphisms, since these transformations must be given by $\mathcal{C}^{\infty}$ functions. As in Theorem 6.4, we need to consider separately the cases $a \neq 0$ and $a = 0$.

First, suppose that $a \neq 0$. Then

$$\begin{aligned} C(x, t, p, q) &= 2q\left[apt - u_q(x, q)\,p + u_x(x, q)\,t + \frac{1}{a}\left[u_{xq}(x, q) - u_x(x, q)\,u_q(x, q)\right]\right] \\ &= 2q\left[a\left(p + \frac{u_x}{a}\right)\left(t - \frac{u_q}{a}\right) + \frac{u_{xq}}{a}\right]. \end{aligned}$$

The condition that C is $\mathcal{C}^\infty$, but not identically vanishing, along $q = 0$ forces

$$u(x, q) = -ab \ln |q| + \tilde{u}(x, q)$$

for some nonzero constant b and some $\mathcal{C}^\infty$ function $\tilde{u}(x, q)$. (This can be seen by examining the first-order Taylor expansion of C at $q = 0$: one sees immediately that u_x, u_q, and u_{xq} can each differ from a $\mathcal{C}^\infty$ function only by terms involving times q^{-1}, and that the corresponding $\ln |q|$ term in u must be independent of x.)

By applying an F-translation with $F = -\frac{\tilde{u}(x,q)}{a}$, we can transform $\tilde{u}(x, q)$ to zero, and so C becomes

$$C(x, t, p, q) = 2q \left[ap \left(t + \frac{b}{q} \right) \right] = 2ap\,(tq + b).$$

Product diffeomorphisms leave the constants a and b unchanged, so these are invariants of the system (M, ω, Ω). Calculations analogous to those in Theorem 6.4 show that the GIEs have the stated form.

Next, suppose that $a = 0$. Then

$$C(x, t, p, q) = 2q \left[-u_q(x, q)\, p + u_x(x, q)\, t + \tilde{Q}(x, q) \right],$$

where

$$u_{xq} = u_x u_q$$

and

$$\tilde{Q}(x, q) = \frac{c_0(x)}{2q} + \bar{Q}(x, q)$$

for some $\mathcal{C}^\infty$ functions $c_0(x), \bar{Q}(x, q)$. As in Theorem 6.4, we have

$$u(x, q) = -\ln |\phi(q) + \psi(x)|$$

where we assume without loss of generality that $\phi(0) = 0$, and with $\phi'(q) \neq 0$, $\psi'(x) \neq 0$ on the hyperbolic locus. (It is, however, possible that $\phi'(0) = 0$.) Since $\psi'(x) \neq 0$, it follows that u cannot be singular at every point of the set $\{q = 0\}$, and therefore $c_0(x)$ must be nonzero.

By applying a product diffeomorphism, we can transform $\psi(x)$ to $x + 1$. Then we have

$$C(x, t, p, q) = 2q \left[\frac{\phi'(q)p - t}{\phi(q) + x + 1} + \frac{c_0(x)}{2q} + \bar{Q}(x, q) \right].$$

By applying an F-translation, where F is a solution of

$$F_{xq} + \frac{\phi'(q)F_x - F_q}{\phi(q) + x + 1} + \bar{Q} = 0$$

in a neighborhood of e, we can transform $\bar{Q}$ to zero. Finally, we have

$$C(x, t, p, q) = \frac{2q\,(\phi'(q)p - t)}{\phi(q) + x + 1} + c_0(x).$$

Note that if this equation had a zero locus, it would be given by

$$\mathcal{Z} = \{q = c_0(x) = 0\}.$$

This contradicts our hypotheses regarding $\mathcal{Z}$; therefore, the function $c_0(x)$ must be nonvanishing, and $\mathcal{Z}$ is empty. Calculations analogous to those in Theorem 6.4 show that the GIEs have the stated form. □

Next we consider the analog of Theorem 6.5 in the involutive type-changing case. Because $V^\perp$ is singled out as being the characteristic subsystem containing the CIE that defines the type-changing locus, there are two versions of this theorem, depending on whether $V^\perp$ or $\overline{V}^\perp$ is the completely integrable subsystem. The case of Theorem 7.7 will be of interest to us in [CKW].

Theorem 7.7. *Let (M, ω, Ω) be involutive type-changing of order 1 with CIEs, and suppose that $V^\perp$ contains at least one GIE and $\overline{V}^\perp$ is completely integrable. Then at any point e in $\mathcal{P}$ or $\mathcal{Z}$, (M, ω, Ω) is locally symplectically equivalent to (7.1), where e corresponds to $(0,0,1,0)$ if $e \in \mathcal{P}$ and to $(0,0,0,0)$ if $e \in \mathcal{Z}$, and $C_p = 2b$ for an invariant real, nonzero constant b. Specifically:*

(1) If b is not a negative integer, then (M, ω, Ω) has local normal form (7.1) with

$$C = 2bp.$$

The CIEs belonging to $\overline{V}^\perp$ are represented by the 1-form $d\bar{h}$, where

$$\bar{h} = \bar{h}(x,\, pq^b)$$

and $\bar{h}$ is an arbitrary $\mathcal{C}^\infty$ function of two variables. The GIEs belonging to $V^\perp$ are represented by the 1-form θ_h, where

$$h = \left(\frac{b+1}{b}\right) tq + k(q)$$

and k is an arbitrary $\mathcal{C}^\infty$ function of one variable.

(2) If b is a negative integer, then (M, ω, Ω) has either the local normal form above, or the local normal form (7.1) with

$$C = 2bp + 2c(x)q^{-b}$$

for some $\mathcal{C}^\infty$ function $c(x)$. In the latter case, the CIEs belonging to $\overline{V}^\perp$ are represented by the 1-form $d\bar{h}$, where

$$\bar{h} = \bar{h}(x,\, pq^b + c(x)\ln|q|)$$

and $\bar{h}$ is an arbitrary $\mathcal{C}^\infty$ function of two variables, and the GIEs belonging to $V^\perp$ are represented by the 1-form θ_h, where

$$h = \left(\frac{b+1}{b}\right) tq - \left(\frac{1}{b}\right) q^{-b} \int c(x)\, dx + k(q)$$

and k is an arbitrary $\mathcal{C}^\infty$ function of one variable.

Remark 7.8. Note that the CIEs in Case (2) above do not extend smoothly to the parabolic locus.

Proof. Choose local coordinates so that Ω has the normal form (7.1) with $m = 1$. An analogous argument to that given in the proof of Theorem 6.5 shows that

$$C(x,t,p,q) = -\frac{2p}{\beta(q)} + \widetilde{C}(x,q), \tag{7.2}$$

where $\beta(q)$ is a $\mathcal{C}^\infty$ function which satisfies $\beta(0) \neq 0$.

The effect of a product diffeomorphism is more subtle than that in Theorem 6.5. In order to preserve the singular locus $\mathcal{P} \cup \mathcal{Z}$, we must restrict to diffeomorphisms $\mathcal{Q}(q)$ satisfying $\mathcal{Q}(0) = 0$, and under such a product diffeomorphism $(\chi(x), \mathcal{Q}(q))$ with $\chi(x) = x$, we have

$$\beta(q) \mapsto \frac{\mathcal{Q}(q)}{q\mathcal{Q}'(q)}\beta(\mathcal{Q}(q)).$$

In particular, we can write

$$\mathcal{Q}(q) = cq + q^2\bar{\mathcal{Q}}(q)$$

for some nonzero constant c and $\mathcal{C}^\infty$ function $\bar{\mathcal{Q}}(q)$, and then we have

$$\beta(0) \mapsto \left(\left.\frac{c + q\bar{\mathcal{Q}}(q)}{\bar{\mathcal{Q}}'(q)}\right|_{q=0} \right) \beta(0) = \left(\frac{c}{c}\right)\beta(0) = \beta(0).$$

Thus, the value $\beta(0)$ is invariant under product diffeomorphisms.

Now, set $\bar{\beta}(q) = \frac{\beta(q)}{\beta(0)}$, and consider the initial value problem

$$q\mathcal{Q}'(q) = \mathcal{Q}(q)\bar{\beta}(\mathcal{Q}(q)), \qquad \mathcal{Q}(0) = 0. \tag{7.3}$$

Lemma 7.9. *The initial value problem* (7.3) *has a $\mathcal{C}^\infty$ solution $\mathcal{Q}(q)$ in a neighborhood of $q = 0$.*

Proof of Lemma. By hypothesis, we can write

$$\frac{1}{\bar{\beta}(q)} = 1 + q\bar{\beta}_1(q)$$

for some $\mathcal{C}^\infty$ function $\bar{\beta}_1(q)$. Then equation (7.3) can be written as

$$\frac{\mathcal{Q}'(q)}{\mathcal{Q}(q)} + \bar{\beta}_1(\mathcal{Q}(q))\mathcal{Q}'(q) = \frac{1}{q}.$$

Integrating yields

$$\ln \mathcal{Q}(q) + \beta_1(\mathcal{Q}(q)) = \ln q + k,$$

where $\beta_1(q) = \int \bar{\beta}_1(q)\, dq$. Exponentiating yields

$$\mathcal{Q}(q)e^{\beta_1(\mathcal{Q}(q))} = Kq$$

for some nonzero constant K, and this equation clearly has a $\mathcal{C}^\infty$ solution $\mathcal{Q}(q)$ in a neighborhood of $q = 0$. $\square$

Now apply a product diffeomorphism with $\chi(x) = x$ and $\mathcal{Q}(q)$ a solution of (7.3); this transforms C to the normal form

$$C(x,t,p,q) = 2bp + \widetilde{C}(x,q)$$

where $b = -\frac{1}{\beta(0)} \neq 0$. (The transformed function $\widetilde{C}(x,q)$ may differ from that in (7.2).)

Next, we compute that under an F-translation, we have

$$\widetilde{C}(x,q) \mapsto \widetilde{C}(x,q) + 2qF_{xq} + 2bF_x.$$

So if we could solve the equation

$$qF_{xq} + bF_x = -\tfrac{1}{2}\widetilde{C}(x,q) \tag{7.4}$$

for a $\mathcal{C}^\infty$ function $F(x,q)$, we could transform $\widetilde{C}(x,q)$ to zero via an F-translation. The *formal* solution of (7.4) is

$$F(x,q) = -\tfrac{1}{2}\int \left[q^{-b} \int q^{b-1}\widetilde{C}(x,q)\,dq \right] dx.$$

However, this function is not necessarily $\mathcal{C}^\infty$ in all cases. In particular, consider the $(N-1)$th-order Taylor expansion of $\widetilde{C}(x,q)$ in terms of q, where $N > |b| + 1$:

$$\widetilde{C}(x,q) = \sum_{k=0}^{N-1} c_k(x) q^k + q^N \widetilde{C}_N(x,q),$$

where $\widetilde{C}_N(x,q)$ is $\mathcal{C}^\infty$. If b is a negative integer, then the $c_{-b}(x)q^{-b}$ term in this expansion will give rise to a term in F_x which is not $\mathcal{C}^\infty$. The following lemma shows how to resolve this issue.

Lemma 7.10. *The differential equation*

$$qF_{xq} + bF_x = -\tfrac{1}{2}\widehat{C}(x,q), \tag{7.5}$$

has a $\mathcal{C}^\infty$ solution $F(x,q)$ in a neighborhood of $(x,q) = (0,0)$, where

$$\widehat{C}(x,q) = \begin{cases} \widetilde{C}(x,q) - c_{-b}(x)q^{-b} & \text{if } b \text{ is a negative integer} \\ \widetilde{C}(x,q) & \text{otherwise.} \end{cases}$$

Proof of Lemma. Choose $N > |b| + 1$, and write

$$\widehat{C}(x,q) = \sum_{k=0}^{N-1} c_k(x) q^k + q^N \widehat{C}_N(x,q), \tag{7.6}$$

where $\widehat{C}_N(x,q)$ is $\mathcal{C}^\infty$. By hypothesis, we have $c_{-b}(x) \equiv 0$ if b is a negative integer.

By applying an F-translation with

$$F(x,q) = \int \left[\sum_{k=0, k\neq -b}^{N-1} -\frac{c_k(x)q^k}{2(k+b)} \right] dx,$$

we can transform the leading terms of $\widehat{C}(x,q)$ to zero; i.e., we can assume that

$$\widehat{C}(x,q) = q^N \widehat{C}_N(x,q).$$

Now the formal solution

$$F(x,q) = -\tfrac{1}{2} \int \left[q^{-b} \int_0^q \lambda^{N+b-1} \widehat{C}_N(x,\lambda)\, d\lambda \right] dx$$

is in fact a smooth solution of (7.5), as desired. □

Lemma 7.10 shows that by performing an F-translation, we can arrange that:

- $C(x,t,p,q) = 2bp$ if b is not a negative integer.
- $C(x,t,p,q) = 2bp + 2c(x)q^{-b}$ for some $\mathcal{C}^\infty$ function $c(x)$ if b is a negative integer.

Calculations analogous to those in Theorem 6.4 show that the CIEs and GIEs have the stated form. □

Remark 7.11. In Case (2) above, a product diffeomorphism $(\chi(x), \mathcal{Q}(q))$ with $\mathcal{Q}(q) = q$ transforms the function $c(x)$ to $\chi'(x)c(\chi(x))$. Thus, if $c(0) \neq 0$ (or if $c(x)$ vanishes to finite order n at $x = 0$), we can apply such a product diffeomorphism to arrange that $c(x) = 1$ (or that $c(x) = x^n$).

Theorem 7.12. *Let (M, ω, Ω) be involutive type-changing of order 1 with CIEs, and suppose that $V^\perp$ is completely integrable and $\overline{V}^\perp$ contains at least one GIE. Then $\mathcal{Z}$ is empty, and at any point $e \in \mathcal{P}$, (M, ω, Ω) is locally symplectically equivalent to (7.1), where e corresponds to $(0,0,1,0)$ and*

$$C = 2tq + c_0(x)$$

for some nonvanishing function $c_0(x)$. The CIEs belonging to $\overline{V}$ are represented by the 1-form dh, where

$$h = h\big(q,\, 2tqe^{-x} - \int c_0(x) e^{-x}\, dx\big)$$

and h is an arbitrary $\mathcal{C}^\infty$ function of two variables. The GIEs belonging to $V^\perp$ are represented by the 1-form $\theta_{\bar{h}}$, where

$$\bar{h} = tq + p + \tfrac{1}{2} c_0(x) \ln|q| + \bar{k}(x)$$

and $\bar{k}$ is an arbitrary $\mathcal{C}^\infty$ function of one variable.

Remark 7.13. Note that the GIEs above do not extend smoothly to the parabolic locus.

Proof. Choose local coordinates so that Ω has the normal form (7.1) with $m = 1$. An analogous argument to that given in the proof of Theorem 6.5 – with the roles of $V^\perp$ and $\overline{V}^\perp$ reversed – shows that

$$C(x,t,p,q) = -\frac{2tq}{\bar{\beta}(x)} + \widetilde{C}(x,q), \tag{7.7}$$

where $\bar{\beta}(x)$ is a $\mathcal{C}^\infty$ function which satisfies $\bar{\beta}(0) \neq 0$.

Under a product diffeomorphism $(\chi(x), \mathcal{Q}(q))$ with $\mathcal{Q}(q) = q$, we have

$$\bar{\beta}(x) \mapsto \frac{1}{\chi'(x)} \bar{\beta}(\chi(x)).$$

By applying such a product diffeomorphism with $\chi(x)$ a solution of

$$\chi'(x) = \bar{\beta}(\chi(x)),$$

we can arrange that $\bar{\beta}(x) \equiv 1$ in a neighborhood of $x = 0$.

Next, we compute that under an F-translation, we have

$$\widetilde{C}(x, q) \mapsto \widetilde{C}(x, q) + 2q(F_{xq} - F_q).$$

We can write

$$\widetilde{C}(x, q) = c_0(x) + q\widetilde{C}_1(x, q)$$

for some $\mathcal{C}^\infty$ function $\widetilde{C}_1(x, q)$, and by performing an F-translation with F a solution to

$$F_{xq} - F_q + \tfrac{1}{2}\widetilde{C}_1(x, q) = 0,$$

we can transform $\widetilde{C}_1(x, q)$ to zero. We now have

$$C(x, t, p, q) = 2tq + c_0(x),$$

as desired. The same argument as that given at the end of the proof of Theorem 7.5 shows that the function $c_0(x)$ must be nonvanishing, and that therefore $\mathcal{Z}$ is empty. Calculations analogous to those in Theorem 6.4 show that the CIEs and GIEs have the stated form. □

The following theorem extends the classical Lie-Darboux normal form of Theorem 6.6 to the involutive type-changing setting, and indicates that only one of the Martinet generic normal forms [Mar70] occurs in the context of involutive type-changing symplectic Monge-Ampère PDEs.

Theorem 7.14. *Let (M, ω, Ω) be involutive type-changing of order 1 with CIEs, and suppose that $V^\perp$, $\overline{V}^\perp$ are both completely integrable. Then $\mathcal{Z}$ is empty, and at any point $e \in \mathcal{P}$, (M, ω, Ω) is locally symplectically equivalent to (7.1), where e corresponds to $(0, 0, 1, 0)$ and $C = 2$. The CIEs belonging to $V^\perp$ and $\overline{V}^\perp$ are represented by the 1-forms dh and $d\bar{h}$, respectively, where*

$$h = h(q, tq - x), \qquad \bar{h} = \bar{h}(x, qe^p),$$

and h, $\bar{h}$ are arbitrary $\mathcal{C}^\infty$ functions of two variables.

Proof. Choose local coordinates so that Ω has the form (7.1) with $m = 1$. As in the proof of Theorem 6.6, the complete integrability of $\overline{V}$ and $\overline{V}^\perp$ yields $C_p = C_t = 0$, so that

$$C = C(x, q).$$

Under an F-translation, we have

$$C(x, q) \mapsto C(x, q) + 2qF_{xq}.$$

We can write

$$C(x, q) = c_0(x) + qC_1(x, q)$$

for some $\mathcal{C}^\infty$ function $C_1(x,q)$, and by performing an F-translation with F a solution to

$$F_{xq} + \tfrac{1}{2}C_1(x,q) = 0,$$

we can transform $C_1(x,q)$ to zero. We now have $C = c_0(x)$, and the same argument as that given at the end of the proof of Theorem 7.5 shows that the function $c_0(x)$ must be nonvanishing, and that therefore $\mathcal{Z}$ is empty.

Under a product diffeomorphism $(\chi(x), \mathcal{Q}(q))$ with $\mathcal{Q}(q) = q$, we have

$$c_0(x) \mapsto \chi'(x)c_0(\chi(x)).$$

By applying such a product diffeomorphism with $\chi(x)$ a solution of

$$\chi'(x) = \frac{2}{c_0(\chi(x))},$$

we can arrange that $c_0(x) \equiv 2$ in a neighborhood of $x = 0$. Then we have $C(x,t,p,q) \equiv 2$, as desired. Calculations analogous to those in Theorem 6.4 show that the CIEs have the stated form. $\square$

8. Applications

8.1. **Local existence for ill-posed initial value problems.** Consider the collection of indefinite (formal) functionals

$$\mathcal{L}^s(f) = \int_0^s \int_{\mathbb{R}} L(f_t)\, f_x \, dx\, dt, \tag{8.1}$$

where $z = f(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{E}$ is viewed as a path

$$t \mapsto f(\cdot, t)$$

in the space of $\mathcal{C}^\infty$ functions from $\mathbb{R}$ to the Euclidean line, based at $f_0(x) = f(x,0)$. Here $L(q)$ is a smooth function with isolated zeros. (Note that these functionals are "geometric" in the sense that they are invariant under the groups of diffeomorphisms $x \to \chi(x)$.) The Euler-Lagrange PDE

$$2L'(f_t)f_{xt} + f_x L''(f_t)f_{tt} = 0 \tag{8.2}$$

for the functional (8.1) is a symplectic Monge-Ampère PDE. On the symplectic reduction $M = T^*\mathbb{R}^2$, the natural initial value problem is:

$$p_0(x) = f_x(x,0), \qquad q_0(x) = f_t(x,0).$$

If $L(q)$ has a nondegenerate critical point at $q = q_c$, then (8.2) is involutive type-changing of order 1 with CIEs as in Theorem 7.7, Case (1), with $b = \frac{1}{2}$ and

$$\mathcal{P} \cup \mathcal{Z} = \{q = q_c\}, \qquad \mathcal{Z} = \{q = q_c,\, p = 0\}.$$

More concretely, near such a critical point $q = q_c$, there is a symplectomorphism

$$q \mapsto \mathcal{Q}(q), \qquad t \mapsto \frac{t}{\mathcal{Q}'(q)}$$

which transforms (8.2) to the Euler-Langrange equation for the functional $L(q) = \frac{1}{2}q^2$:

$$2f_t f_{xt} + f_x f_{tt} = 0, \tag{8.3}$$

with initial conditions

$$\bar{p}_0(x) = f_x(x, 0), \qquad \bar{q}_0(x) = f_t(x, 0) \tag{8.4}$$

for $|x| < \epsilon$, $|\bar{p}_0(x)| < \epsilon$, $|\bar{q}_0(x)| < \epsilon$. In [MR98], the authors consider this PDE as the geodesic equation for a natural metric on curves in the space of embeddings $f : \mathbb{R} \to \mathbb{E}$ and construct global, nonsingular solutions.

If $\bar{p}_0(0) = 0$, $\bar{q}_0(0) \neq 0$, then the initial value problem (8.3), (8.4) is ill-posed in the classical sense; i.e., $f_{tt}(x, 0)$ cannot be expressed in terms of the initial conditions. However, this initial data yields *non*-characteristic initial value problems for the GIEs

$$f = 3tf_t + k(f_t),$$

where k is an arbitrary function of one variable. This GIE effectively "repairs" the original initial value problem. On the other hand, if $\bar{q}_0(0) = \bar{p}_0(0) = 0$, then the initial conditions intersect the zero locus $\mathcal{Z}$, and there is no local existence theory for the second-order initial value problem (8.3), (8.4). In [CKW], we will extend these methods to show how the intermediate equations may be used to prove local and global existence for this type of initial value problem. In particular, we will construct global solutions for (8.3) with topologically nontrivial singularites. Note that the parabolic locus does play an indirect role, in that it must interact with the zero locus in order that we have our simple normal form.

These results are not obvious; for some PDEs with noninvolutive zero points, there are obstructions to the existence of such solutions. However, for an involutive type-changing PDE with zero points as in the first part of Theorem 7.7, we have the following consequences of our normal form. First, there exists a transverse foliation of a neighborhood of $\mathcal{Z}$ by Lagrangian solutions, given in the above local coordinates by

$$\{x = x_0, t = t_0\}, \qquad x_0, t_0 \in \mathbb{R}.$$

As an example, the involutive type-changing PDE

$$2(f_t)^m f_{xt} + f_x(f_{tt} - f_{xx}f_{tt} + f_{xt}^2) = 0$$

with $\mathcal{Z} = \{p = q = 0\}$ has the transverse foliation by Lagrangian solutions of the form

$$f(x, t) = (x - x_0)^2 + (t - t_0)^2.$$

We conjecture that this foliation is unique when $m = 1$.

Second, a linearization tensor constructed at the zero points can be used to show that there do *not* exist any local Langrangian solutions $s : D \to M$ which are transverse both to $\mathcal{Z}$ and to the above foliation. This linearization is a Lie derivative construction analogous to the "linear part" of a zero point in a vector field, and suggests that there is a global index theory for symplectic Monge-Ampère equations with zero points.

8.2. **Symmetry groups.** The well-known theorem of Noether guarantees a one-to-one correspondence between symmetries and conservation laws for PDEs which arise as Euler-Lagrange equations for variational problems. (See, e.g., [Lyc79] for a discussion of Noether's theorem.) However, the relationship between symmetries and IDEs for such PDEs is more subtle. For example, the Euler-Lagrange PDEs for the functionals $L(q) = q^m \bar{L}(q)$ and $L(q) = \int e^{\bar{L}(q)}\, dq$, as in §8.1, have no Noether-type correspondence between IDEs and symmetry vector fields of (M, ω, Ω). Furthermore, while the exterior differential system $(\mathbb{R}^4, \omega, \Omega)$ given by (8.3) is smooth on a neighborhood of the origin, the F-translations defined by

$$F(x, q) = \frac{c(x)}{q},$$

with $c(x)$ $\mathcal{C}^\infty$ on a neighborhood of the origin, define an infinite set of symmetries of the EDS which do *not* $\mathcal{C}^\infty$-extend to the origin. These phenomena merit further study, and the normal forms of §6 and §7 should be useful for exploring and clarifying the relationship between symmetries and IDEs for symplectic Monge-Ampère PDEs of Euler-Lagrange type.

References

[BG95a] Robert L. Bryant and Phillip A. Griffiths, *Characteristic cohomology of differential systems. I. General theory*, J. Amer. Math. Soc. **8** (1995), no. 3, 507–596.

[BG95b] ______, *Characteristic cohomology of differential systems. II. Conservation laws for a class of parabolic equations*, Duke Math. J. **78** (1995), no. 3, 531–676.

[Bry95] Robert L. Bryant, *An introduction to Lie groups and symplectic geometry*, Geometry and quantum field theory (Park City, UT, 1991), IAS/Park City Math. Ser., vol. 1, Amer. Math. Soc., Providence, RI, 1995, pp. 5–181.

[Car45] Élie Cartan, *Les systèmes différentiels extérieurs et leurs applications géométriques*, Actualités Sci. Ind., no. 994, Hermann et Cie., Paris, 1945.

[CKW] Jeanne Clelland, Marek Kossowski, and George R. Wilkens, *Constructing topologically distinct energy-critical curves in the path space of the Euclidean line*, to appear in *The Journal of Differential Equations.*

[Cle97a] Jeanne Nielsen Clelland, *Geometry of conservation laws for a class of parabolic partial differential equations*, Selecta Math. (N.S.) **3** (1997), no. 1, 1–77.

[Cle97b] ______, *Geometry of conservation laws for a class of parabolic PDE's. II. Normal forms for equations with conservation laws*, Selecta Math. (N.S.) **3** (1997), no. 4, 497–515.

[Cle00] ______, *On the intermediate integral for Monge-Ampère equations*, Proc. Amer. Math. Soc. **128** (2000), no. 2, 527–531.

[Gou96] Édouard Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes*, vol. Tome 1, Hermann, Paris, 1896.

[IL03] Thomas A. Ivey and J. M. Landsberg, *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, RI, 2003.

[JA97] Martin Juráš and Ian M. Anderson, *Generalized Laplace invariants and the method of Darboux*, Duke Math. J. **89** (1997), no. 2, 351–375.

[Jur96] Martin Juráš, *Generalized Laplace invariants and classical integration methods for second order scalar hyperbolic partial differential equations in the plane*, Differential geometry and applications (Brno, 1995), Masaryk Univ., Brno, 1996, pp. 275–284.

[Kos91] Marek Kossowski, *Local existence of multivalued solutions to analytic symplectic Monge-Ampère equations (the nondegenerate and type changing cases)*, Indiana Univ. Math. J. **40** (1991), no. 1, 123–148.

[Kus92] A. G. Kushner, *Chaplygin and Kel′dysh normal forms of Monge-Ampère equations of variable type*, Mat. Zametki **52** (1992), no. 5, 63–67, English trans. in Math. Notes **52** (1992).

[Kus95] A. Kushner, *Symplectic geometry of mixed type equations*, The interplay between differential geometry and differential equations, Amer. Math. Soc. Transl. Ser. 2, vol. 167, Amer. Math. Soc., Providence, RI, 1995, pp. 131–142.

[Kus98] A. G. Kushner, *Monge-Ampère equations and e-structures*, Dokl. Akad. Nauk **361** (1998), no. 5, 595–596.

[Lep50] Th. H. Lepage, *Sur certains idéaux de l'algèbre extérieure de degré* $2n$, $n > 1$, Algèbre et Théorie des Nombres., Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, Centre National de la Recherche Scientifique, Paris, 1950, pp. 181–186.

[LR90] V. V. Lychagin and Yu. R. Romanovsky, *Nonholonomic intermediate integrals of partial differential equations*, Differential geometry and its applications (Brno, 1989), World Sci. Publ., Teaneck, NJ, 1990, pp. 355–360.

[LRC93] V. V. Lychagin, V. N. Rubtsov, and I. V. Chekalov, *A classification of Monge-Ampère equations*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 3, 281–308.

[Lyc79] V. V. Lychagin, *Contact geometry and second-order nonlinear differential equations*, Russian Math. Surveys **34** (1979), no. 1, 149–180.

[Lyc85] ———, *Topological aspects of the geometric theory of differential equations*, Application of topology in modern analysis (Russian), Novoe Global. Anal., Voronezh. Gos. Univ., Voronezh, 1985, pp. 106–123, transl. in Global analysis – studies and applications, II, Lecture Notes in Math., vol. 1214, Springer, Berlin, 1986, pp. 100-116.

[Mar70] Jean Martinet, *Sur les singularités des formes différentielles*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. fasc. 1, 95–178.

[MR98] P. W. Michor and T. S. Ratiu, *On the geometry of the Virasoro-Bott group*, J. Lie Theory **8** (1998), no. 2, 293–309.

[Zil99] L. V. Zil'bergleĭt, *Intermediate integrals of the Monge-Ampère equations*, Izv. Vyssh. Uchebn. Zaved. Mat. (1999), no. 9, 16–25.

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