One of the attractions of control theory is the interdisciplinary nature of the subject. A glance at the proceedings of the annual *Conference on Decision and Control* indicates the large number of people and departments actively working and collaborating on problems of control. Attending professional meetings with colleagues from engineering, computer science, chemistry and other departments, as well as mathematics departments, is the rule rather than the exception.

The enormous breadth of the subject opens the door to many types of mathematical applications. The origins of control theory result from trying to model real world processes that admit a certain amount of control. These models nearly always take the form of differential equations, and are usually an underdetermined system of ordinary differential equations. The starting point for most people is the theory of linear control systems. These are systems of the form

$$\frac{dx}{dt} = A\,x + B\,u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{1}$$

where $A$ and $B$ are constant matrices and $m < n$. A classic introduction to the theory of linear systems is Brockett's book [1].

A great deal of the current research focuses on nonlinear control systems. Control theorist quickly discovered that modern differential geometry provides the best setting to analyze and solve a number of problems. Isidori's text on nonlinear control systems [8] begins with the notions of vector fields, Lie brackets and distributions on manifolds.

The subject of geometric control theory developed quite rapidly, but largely ignored techniques based on differential forms and exterior differential systems until 1983 when Gardner [2] first suggested applying Cartan's method of equivalence. Some years later Gardner and Shadwick [6] would revisit a famous problem and provide the optimal solution. The problem is to determine necessary and sufficient conditions on a nonlinear control system in order that it be transformable to a linear system by an admissible change of variables. For this problem the admissible changes of variables are the *feedback transformations*. A feedback transformation is a change of variables of the form

$$\begin{aligned} \bar{x} &= \varphi(x) \\ \bar{u} &= \psi(x, u). \end{aligned} \tag{2}$$

This problem was solved by van der Schaft [16] in 1984.

Using techniques from exterior differential systems, Gardner and Shadwick made a substantial improvement on this earlier result. Every solution to this problem uncovers a collection of completely integrable Pfaffian systems. Finding the change of variables that linearizes the control system thus requires integrating each of these Pfaffian systems. The great improvement made by Gardner and Shadwick is that their technique is guaranteed to give the best possible collection of Pfaffian systems: they will be smallest in number and in dimension. Integrating these systems provides part of the required change of variables, and the remaining coordinates are determined by differentiation and linear algebra.

Another group of control theorist, based at Berkeley and Cal Tech, have been making enormous progress on problems involving the control of multi-trailer systems, such as the baggage carts that deliver luggage between an airplane and the terminal. At the core of their work is an exterior differential system result know as the Goursat normal form. Using this result, they are able to find a good set of coordinates to use. With these coordinates and some very clever use of sinusoids they are able to back a multi trailer system into a loading bay or even parallel park such a system. These applications as well as numerous extensions can be found in references [3, 4, 9, 10, 11, 17, 12, 13, 14, 15].

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