GEOMETRY OF SUB-FINSLER ENGEL MANIFOLDS

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ABSTRACT. We analyze the geometry of sub-Finsler Engel manifolds, computing a complete set of local invariants for a large class of these manifolds. We derive geodesic equations for regular geodesics and show that in the symmetric case, the rigid curves are local minimizers. We end by illustrating our results with an example.

1. INTRODUCTION

In [3], the first two authors defined the notion of *sub-Finsler geometry* as a natural generalization of sub-Riemannian geometry and undertook a detailed analysis of sub-Finsler contact 3-manifolds. In this paper we continue the study of sub-Finsler geometry by analyzing the class of sub-Finsler Engel manifolds; this is a natural case to consider next, since the generic rank 2 distribution on a 4-manifold forms an Engel structure. Engel manifolds present a more complicated picture than contact 3-manifolds, due largely to the presence of so-called *rigid curves*. Moreover, the class of Engel manifolds contains many examples that arise naturally in control theory, perhaps most notably the "penny on the plane" problem.

First, recall some definitions:

Definition 1.1. *An* Engel manifold *is a 4-manifold $\mathcal{X}$ equipped with a rank 2 distribution D satisfying the conditions that*
- $D^{(1)} = D + [D, D]$ *has rank 3, and*
- $D^{(2)} = D^{(1)} + [D^{(1)}, D^{(1)}] = T\mathcal{X}$

at each point of $\mathcal{X}$.

Definition 1.2. *A curve $\gamma : [a, b] \to \mathcal{X}$ is a* horizontal curve *of the distribution D on $\mathcal{X}$ if $\gamma'(t) \in D$ whenever $\gamma'(t)$ exists.*

Definition 1.3. *A horizontal curve $\gamma : [a, b] \to \mathcal{X}$ is called* regular *if there exists a C^1, 1-parameter family $\Gamma : [a, b] \times (-\varepsilon, \varepsilon) \to \mathcal{X}$ of horizontal curves $\gamma_s = \Gamma(\cdot, s) : [a, b] \to \mathcal{X}$ such that:*

- *$\gamma_0 = \gamma$.*
- *$\forall s, \gamma_s(a) = \gamma(a)$ and $\gamma_s(b) = \gamma(b)$; i.e., Γ is an endpoint-preserving variation.*
- *The vector field $\frac{d}{ds}\big|_{s=0} \Gamma$ is linearly independent from $\gamma'(t)$ for some $t \in [a, b]$; i.e., the curves γ_s are not merely reparametrizations of γ.*

If no such variation exists, then γ is called rigid.

Note that any sub-curve of a rigid curve is rigid, while any horizontal extension of a regular curve is regular. However, it is possible that a rigid curve may become regular upon extension.

Definition 1.4. *A horizontal curve* $\gamma : [a, b] \to \mathfrak{X}$ *is called* locally rigid *if, for every* $t \in (a, b)$, *there exists* $\varepsilon > 0$ *such that the restriction* $\gamma : (t - \varepsilon, t + \varepsilon) \to \mathfrak{X}$ *is a rigid curve.*

An important feature of an Engel structure $(\mathfrak{X}, D)$ is the set of *rigid directions*. These may be described as follows: let $\bar{V}_1, \bar{V}_2$ be linearly independent vector fields on a suitably chosen neighborhood $\mathcal{U} \subset \mathfrak{X}$ which span $D_x \subset T_x\mathfrak{X}$ at each point $x \in \mathcal{U}$. (The reason for the bars will become apparent in §3.) Let

$$\bar{V}_3 = -[\bar{V}_1, \bar{V}_2];$$

since D is an Engel distribution, $\bar{V}_3$ is linearly independent from $\bar{V}_1$ and $\bar{V}_2$, and the rank 3 distribution $D^{(1)}$ spanned by $\bar{V}_1, \bar{V}_2, \bar{V}_3$ at each point (called the *first derived system* of D) is well-defined, independent of the choice of $\bar{V}_1, \bar{V}_2$.

Now, for each $x \in \mathcal{U}$, consider the map

$$\phi_x : D_x \to T_x\mathfrak{X}/D_x^{(1)}$$

given by

$$\phi_x(W) = [W, \bar{V}_3]_x.$$

Since D is an Engel distribution, ϕ_x is a surjective linear map; thus it has a 1-dimensional kernel.

Definition 1.5. *Any nonzero vector* $W \in \ker \phi_x$ *is called a* rigid direction *at* $x \in \mathfrak{X}$.

The rigid directions form a smooth line field $L \subset D$ on $\mathfrak{X}$. In [2], Bryant and Hsu prove the following theorem:

Theorem 1.6 (Bryant-Hsu). *Let* $\mathfrak{X}$ *be a 4-manifold equipped with an Engel distribution* D. *Then there is a canonical foliation* $\mathcal{F}$ *of* $\mathfrak{X}$ *by horizontal curves for* D *with the property that any horizontal curve* $\gamma : [a, b] \to \mathfrak{X}$ *is locally rigid if and only if its image lies in a single leaf of* $\mathcal{F}$. *Moreover, this foliation consists precisely of the integral curves of the line field determined by the rigid directions of the Engel structure.*

In particular, this means that there is a rigid curve passing through every point of $\mathfrak{X}$, so these curves are not at all rare.

Definition 1.7. *A* sub-Finsler metric *on an* n-*dimensional manifold* $\mathfrak{X}$ *with a smooth, rank* s *distribution* D *is a smoothly varying Finsler metric on each subspace* $D_x \subset T_x\mathfrak{X}$. *A sub-Finsler manifold, denoted by the triple* $(\mathfrak{X}, D, F)$, *is a smooth* n-*dimensional manifold* $\mathfrak{X}$ *equipped with a sub-Finsler metric* F *on a bracket-generating distribution* D *of rank* $s > 0$. *The* length *of a horizontal curve* $\gamma : [a, b] \to \mathfrak{X}$ *is*

$$\mathcal{L}(\gamma) = \int_a^b F(\gamma'(t)) \, dt.$$

We will be most interested in sub-Finsler Engel manifolds that describe optimal control problems, as in Example 1.8 below. For such control problems, the primary goal is to describe the *optimal trajectories*; these correspond to geodesics of the associated sub-Finsler manifold. Locally rigid curves in the Engel manifold correspond to *abnormal trajectories* of the associated control problem, and these trajectories are generally of significant interest. In particular, the question of whether, and when, the abnormal trajectories are optimal is an important and difficult one. The corresponding question for the associated sub-Finsler

structure is whether, and when, the rigid curves are actually geodesics, in the sense of being length-minimizing curves. We will address this question in §5.

The following example illustrates these concepts nicely:

Example 1.8. ("Kinematic penny on a plane") Consider a wheel of radius 1 rolling without slipping on the Euclidean plane $\mathbb{E}^2$. The wheel's configuration can be represented by the vector ${}^t(x, y, \varphi, \psi)$, where (x, y) is the wheel's point of contact with the plane, φ is the angle of rotation of a marked point on the wheel from the vertical (think of the marked point as Lincoln's head on the penny), and ψ is the wheel's heading angle, i.e., the angle made by the tangent line to the curve traced by the wheel on the plane with the x-axis. Thus the state space has dimension four and is naturally isomorphic to $\mathbb{R}^2 \times S^1 \times S^1$.

The condition that the wheel rolls without slipping is equivalent to the statement that its path ${}^t(x(t), y(t), \varphi(t), \psi(t))$ in the state space satisfies the differential equation

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \\ \dot{\psi} \end{bmatrix} = \dot{\varphi} \begin{bmatrix} \cos\psi \\ \sin\psi \\ 1 \\ 0 \end{bmatrix} + \dot{\psi} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
$$

where the functions $\dot{\varphi}(t), \dot{\psi}(t)$ are control functions which may be specified arbitrarily. The function $\dot{\varphi}$ describes the rate at which the wheel is propelled forward, while the function $\dot{\psi}$ describes how fast the heading angle is rotated. Thus the velocity vector ${}^t(\dot{x}, \dot{y}, \dot{\varphi}, \dot{\psi})$ of any solution curve must lie in the distribution D spanned by the vector fields

$$
\bar{V}_1 = (\cos\psi)\frac{\partial}{\partial x} + (\sin\psi)\frac{\partial}{\partial y} + \frac{\partial}{\partial \varphi}
$$

$$
\bar{V}_2 = \frac{\partial}{\partial \psi}.
$$

It is straightforward to compute that

$$
\bar{V}_3 = -[\bar{V}_1, \bar{V}_2] = -(\sin\psi)\frac{\partial}{\partial x} + (\cos\psi)\frac{\partial}{\partial y}
$$

$$
\bar{V}_4 = -[\bar{V}_2, \bar{V}_3] = (\cos\psi)\frac{\partial}{\partial x} + (\sin\psi)\frac{\partial}{\partial y},
$$

from which we conclude that D is an Engel distribution. (The reason for the minus signs will become apparent later.) Moreover, since $[\bar{V}_1, \bar{V}_3] = 0$, the rigid direction at each point is spanned by $\bar{V}_1$. Therefore, the locally rigid curves are the integral curves of $\bar{V}_1$; these curves correspond to rolling the penny along a straight-line path in the plane.

That these curves have no nontrivial endpoint-preserving variations may be seen directly as follows: let $\gamma : [a, b] \to \mathcal{X}$ be the rigid curve with parametrization

$$
\gamma(t) = (x(t), y(t), \varphi(t), \psi(t)) = \big((\cos\psi_0)(t - t_0), (\sin\psi_0)(t - t_0), t - t_0, \psi_0\big).
$$

Let $\Gamma : [a, b] \times (-\varepsilon, \varepsilon) \to \mathcal{X}$ be any endpoint-preserving variation of γ, and consider the projections $\bar{\gamma}_s$ of the curves γ_s to the xy-plane. First, observe that $\bar{\gamma}_0$ is the unique geodesic joining the points $\bar{\gamma}_0(a)$ and $\bar{\gamma}_0(b)$ in the Euclidean plane. Every curve $\bar{\gamma}_s$ in this family must have the same endpoints, and it follows that if some curve $\bar{\gamma}_s$ is not a reparametrization of $\bar{\gamma}_0$, then it must have length strictly greater than that of $\bar{\gamma}_0$. But then the values of φ – which measures the length of the projection $\bar{\gamma}$ – at the endpoints of γ_0 and γ_s cannot

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agree, and Γ cannot be an endpoint-preserving variation. Therefore, each curve $\bar{\gamma}_s$ must be a reparametrization of $\bar{\gamma}_0$, and it follows easily that each curve γ_s must be a reparametrization of γ_0.

A natural *sub-Riemannian* metric on $(\mathfrak{X}, D)$ may be obtained by declaring the vector fields $\bar{V}_1$, $\bar{V}_2$ to be orthonormal, i.e., by setting

$$\langle \dot{\varphi} \bar{V}_1 + \dot{\psi} \bar{V}_2, \, \dot{\varphi} \bar{V}_1 + \dot{\psi} \bar{V}_2 \rangle = \dot{\varphi}^2 + \dot{\psi}^2.$$

The integral of this quadratic form measures the work done in rotating the heading angle ψ at the rate $\dot{\psi}$ and propelling the wheel forward at the rate $\dot{\varphi}$. With this metric, it is straightforward to show that the rigid curves are, in fact, length-minimizing geodesics. In §6, we will describe a natural sub-Finsler metric on $(\mathfrak{X}, D)$ and see how it compares to this sub-Riemannian one.

This example may be generalized to the case of a penny rolling on any Riemannian surface. In the general case, the locally rigid curves correspond to rolling the penny along a geodesic of the surface. Here we see how a locally rigid curve γ may become regular upon extension: once the path of the penny is long enough to include a pair of conjugate points in its interior, it becomes possible to vary the projection $\bar{\gamma}$ while preserving both the length of $\bar{\gamma}$ and the position and orientation of the penny at the endpoints of $\bar{\gamma}$. Such a variation of $\bar{\gamma}$ may be achieved by an endpoint-preserving variation of γ through horizontal curves of $\mathfrak{X}$.

The remainder of the paper is organized as follows. In §2, we will review the equivalence problem for sub-Riemannian Engel manifolds. In §3, we will use Cartan's method of equivalence to construct a complete set of local invariants for sub-Finsler Engel manifolds. This construction will require a mild assumption on "how non-Riemannian" the sub-Finsler structure can be; we will call sub-Finsler structures which satisfy this condition *tame* sub-Finsler structures. In §4 we derive the geodesic equations for regular geodesics; in §5 we consider separately the issue of rigid geodesics. We conclude by describing an example in §6.

2. Review of sub-Riemannian Engel manifolds

This section is based on the second author's Ph.D. thesis [6].

Suppose that an Engel distribution $(\mathfrak{X}, D)$ is equipped with a *sub-Riemannian metric*, i.e., a Riemannian metric $\langle, \rangle$ on each distribution 2-plane D_x.

We can define a *framing* $\bar{\mathbf{V}} = (\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4)$ – i.e., a set of tangent vector fields which form a basis for $T_x \mathfrak{X}$ at each point – as follows:

- Let $\bar{V}_1 \in D$ be a smooth unit vector field which spans the line field of rigid directions at each point $x \in \mathfrak{X}$. (Note that $\bar{V}_1$ is determined up to sign.)
- Let $\bar{V}_2 \in D$ be a smooth unit vector field which is orthogonal to $\bar{V}_1$ at each point $x \in \mathfrak{X}$. ($\bar{V}_2$ is also determined up to sign.)
- Let $\bar{V}_3 = -[\bar{V}_1, \bar{V}_2]$.
- Let $\bar{V}_4 = -[\bar{V}_2, \bar{V}_3]$.

A framing satisfying these conditions will be called an *adapted framing* for the sub-Riemannian structure on $(\mathfrak{X}, D)$. An adapted framing is unique up to the choice of signs for $\bar{V}_1$ and $\bar{V}_2$; thus the *frame bundle* $\mathcal{B} \to \mathfrak{X}$ defined by

$$\mathcal{B} = \{(x; \bar{\mathbf{V}}_x) | x \in \mathfrak{X} \text{ and } \bar{\mathbf{V}} \text{ is an adapted framing}\}$$

is a principal fiber bundle over $\mathcal{X}$ with discrete fiber group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. There is a *canonical* adapted framing on $\mathcal{B}$, which we will also denote by $\bar{\mathbf{V}} = (\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4)$, given by lifting these vector fields to $\mathcal{B}$ in the obvious way.

Now let $\bar{\mathbf{V}}$ be any adapted framing on $\mathcal{X}$, and consider its *dual coframing*. This is the unique set $\bar{\Omega} = (\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{\omega}^4)$ of linearly independent 1-forms on $\mathcal{X}$ with the property that

$$\bar{\omega}^i(\bar{V}_j) = \delta^i_j, \qquad i, j = 1, \ldots, 4.$$

Such a coframing will be called an *adapted coframing* on $\mathcal{X}$. An adapted coframing is unique up to the same $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action as that for adapted framings, and there is a canonical adapted coframing on $\mathcal{B}$, which will also be denoted by $\bar{\Omega} = (\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{\omega}^4)$, given by lifting these 1-forms to $\mathcal{B}$ in the obvious way.

The Lie bracket equations $[\bar{V}_1, \bar{V}_2] = -\bar{V}_3$, $[\bar{V}_2, \bar{V}_3] = -\bar{V}_4$, together with the condition that $[\bar{V}_1, \bar{V}_3] \in D^{(1)}$, are equivalent to the following equations for the exterior derivatives of the $\bar{\omega}^i$, which are called the *structure equations* of the adapted coframing $\bar{\omega}$ for the sub-Riemannian structure. These may be regarded as either equations for an arbitrary adapted coframing on $\mathcal{X}$ or for the canonical adapted coframing on $\mathcal{B}$:

$$
\begin{aligned}
\text{(2.1)} \quad
d\bar{\omega}^1 &= F^1_{13}\,\bar{\omega}^1 \wedge \bar{\omega}^3 + (F^1_{14}\,\bar{\omega}^1 + F^1_{24}\,\bar{\omega}^2 + F^1_{34}\,\bar{\omega}^3) \wedge \bar{\omega}^4 \\
d\bar{\omega}^2 &= F^2_{13}\,\bar{\omega}^1 \wedge \bar{\omega}^3 + (F^2_{14}\,\bar{\omega}^1 + F^2_{24}\,\bar{\omega}^2 + F^2_{34}\,\bar{\omega}^3) \wedge \bar{\omega}^4 \\
d\bar{\omega}^3 &= \bar{\omega}^1 \wedge \bar{\omega}^2 + F^3_{13}\,\bar{\omega}^1 \wedge \bar{\omega}^3 + (F^3_{14}\,\bar{\omega}^1 + F^3_{24}\,\bar{\omega}^2 + F^3_{34}\,\bar{\omega}^3) \wedge \bar{\omega}^4 \\
d\bar{\omega}^4 &= \bar{\omega}^2 \wedge \bar{\omega}^3 + (F^4_{14}\,\bar{\omega}^1 + F^4_{24}\,\bar{\omega}^2 + F^4_{34}\,\bar{\omega}^3) \wedge \bar{\omega}^4,
\end{aligned}
$$

where the F^i_{jk} are functions on either $\mathcal{X}$ or $\mathcal{B}$, as appropriate, determined by the Lie brackets $[\bar{V}_j, \bar{V}_k]$ via the relationship

$$[\bar{V}_j, \bar{V}_k] = -F^i_{jk}\bar{V}_i.$$

These functions are called *torsion functions*, and they are the local invariants of the sub-Riemannian structure. Computing $d(d\bar{\omega}^4) \equiv 0 \mod \bar{\omega}^4$ yields the single relation

$$F^4_{14} = F^3_{13}$$

among the torsion functions; further differentiation of the structure equations yields only relations among the derivatives of the F^i_{jk}.

3. The sub-Finsler equivalence problem

Let $(\mathcal{X}, D, F)$ be a sub-Finsler manifold consisting of a 4-dimensional manifold $\mathcal{X}$, an Engel distribution D on $\mathcal{X}$, and a sub-Finsler metric F on D. As is the case for Finsler metrics, the sub-Finsler metric F is completely determined by its indicatrix bundle

$$\Sigma = \{\mathbf{u} \in D \mid F(\mathbf{u}) = 1\}.$$

Σ has dimension 5, and each fiber $\Sigma_x = \Sigma \cap D_x$ is a smooth, strictly convex curve in D_x which surrounds the origin $0_x \in D_x$, with the additional condition that the tangent line to Σ_x at each point has contact of precisely order 2 with Σ_x (such a curve is called *strongly convex*). A 5-manifold $\Sigma \subset T\mathcal{X}$ satisfying this condition will be called a *sub-Finsler structure* on $(\mathcal{X}, D)$.

We will compute invariants for sub-Finsler structures via Cartan's method of equivalence. We begin by constructing a coframing on Σ which is nicely adapted to the sub-Finsler structure. In order to make this construction explicit, we will compare a given sub-Finsler structure to a sub-Riemannian structure on $(\mathcal{X}, D)$. This construction is based on the procedure

performed by Bryant in [1] and is similar to that given for sub-Finsler contact 3-manifolds in [3].

Let g be any fixed sub-Riemannian metric on $(\mathcal{X}, D)$, and let Σ_1 be the unit circle bundle for g. Then there exists a well-defined, smooth function $r : \Sigma_1 \to \mathbb{R}^+$ with the property that

$$(3.1) \qquad \Sigma = \{r(\mathbf{u})^{-1}\,\mathbf{u} \mid \mathbf{u} \in \Sigma_1\}.$$

Let $\rho : \Sigma \to \Sigma_1$ be the diffeomorphism which is the inverse of the scaling map defined by r; i.e., ρ satisfies

$$\rho(r(\mathbf{u})^{-1}\,\mathbf{u}) = \mathbf{u}$$

for $\mathbf{u} \in \Sigma_1$.

Let $\pi : \Sigma \to \mathcal{X}$, $\pi_1 : \Sigma_1 \to \mathcal{X}$ denote the respective base point projections, and let $\mathbf{u} \in \Sigma$. (We trust that using the same notation for points in Σ and in Σ_1 will not cause undue confusion.) We will say that a vector $\mathbf{v} \in T_{\mathbf{u}}\Sigma$ is *monic* if $\pi'(\mathbf{u})(\mathbf{v}) = \mathbf{u}$. Since $\pi'(\mathbf{u}) : T_{\mathbf{u}}\Sigma \to T_{\pi(\mathbf{u})}\mathcal{X}$ is surjective with a 1-dimensional kernel, the set of monic vectors in $T_{\mathbf{u}}\Sigma$ is an affine line. A nonvanishing 1-form θ on Σ will be called *null* if $\theta(\mathbf{v}) = 0$ for all monic vectors $\mathbf{v}$, and a 1-form ω on Σ will be called *monic* if $\omega(\mathbf{v}) = 1$ for all monic vectors $\mathbf{v}$. The set of null 1-forms spans a 3-dimensional subspace of $T_{\mathbf{u}}^*\Sigma$ at each point $\mathbf{u} \in \Sigma$, and the difference of any two monic 1-forms is a null form.

3.1. Canonical sub-Riemannian coframing on Σ_1.

In order to effectively compare the sub-Riemannian and sub-Finsler structures, we will lift the canonical coframing for the sub-Riemannian structure on $\mathcal{X}$ to a related coframing on Σ_1.

Let $(\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{\omega}^4)$ be an adapted coframing on $\mathcal{X}$ corresponding to this sub-Riemannian structure. We construct a corresponding adapted coframing $(\alpha, \omega^1, \omega^2, \omega^3, \omega^4)$ on Σ_1 as follows: let $(\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4)$ be vector fields on $\mathcal{X}$ dual to the canonical coframing, and let θ be the coordinate on Σ_1 defined by the property that, for $\mathbf{u} \in \Sigma_1$,

$$\mathbf{u} = (\cos\theta)\bar{V}_1 + (\sin\theta)\bar{V}_2.$$

Then set

$$\alpha = -d\theta$$
$$\omega^1 = (\cos\theta)\,\pi_1^*\bar{\omega}^1 + (\sin\theta)\,\pi_1^*\bar{\omega}^2$$
$$\omega^2 = -(\sin\theta)\,\pi_1^*\bar{\omega}^1 + (\cos\theta)\,\pi_1^*\bar{\omega}^2$$
$$\omega^3 = \pi_1^*\bar{\omega}^3$$
$$\omega^4 = \pi_1^*\bar{\omega}^4.$$

This coframing has the following properties:

(1) ω^1 is a monic form on Σ_1,
(2) ω^2, ω^3, and ω^4 are null forms on Σ_1,
(3) $\pi_1^{-1}D = \{\omega^3, \omega^4\}^\perp$ (Here π_1^{-1} denotes the pullback of the subbundle $D \subset T\mathcal{X}$ to Σ_1).

The structure equations for this coframing are:

$$d\alpha = 0$$

$$d\omega^1 = -\alpha \wedge \omega^2 + (\bar{T}^1_{13}\,\omega^1 + \bar{T}^1_{23}\,\omega^2) \wedge \omega^3 + (\bar{T}^1_{14}\,\omega^1 + \bar{T}^1_{24}\,\omega^2 + \bar{T}^1_{34}\,\omega^3) \wedge \omega^4$$

(3.2) $$\quad d\omega^2 = \alpha \wedge \omega^1 + (\bar{T}^2_{13}\,\omega^1 + \bar{T}^2_{23}\,\omega^2) \wedge \omega^3 + (\bar{T}^2_{14}\,\omega^1 + \bar{T}^2_{24}\,\omega^2 + \bar{T}^2_{34}\,\omega^3) \wedge \omega^4$$

$$d\omega^3 = \omega^1 \wedge \omega^2 + (\bar{T}^3_{13}\,\omega^1 + \bar{T}^3_{23}\,\omega^2) \wedge \omega^3 + (\bar{T}^3_{14}\,\omega^1 + \bar{T}^3_{24}\,\omega^2 + \bar{T}^3_{34}\,\omega^3) \wedge \omega^4$$

$$d\omega^4 = [(\sin\theta)\,\omega^1 + (\cos\theta)\,\omega^2] \wedge \omega^3 + (\bar{T}^4_{14}\,\omega^1 + \bar{T}^4_{24}\,\omega^2 + \bar{T}^4_{34}\,\omega^3) \wedge \omega^4.$$

The functions $\bar{T}^i_{jk}$ appearing in (3.2) can be expressed in terms of the functions F^i_{jk} appearing in (2.1); explicitly:

$$\bar{T}^1_{13} = \cos\theta\left(F^1_{13}\cos\theta + F^2_{13}\sin\theta\right) \qquad \bar{T}^1_{23} = -\sin\theta\left(F^1_{13}\cos\theta + F^2_{13}\sin\theta\right)$$

$$\bar{T}^2_{13} = \cos\theta\left(-F^1_{13}\sin\theta + F^2_{13}\cos\theta\right) \qquad \bar{T}^2_{23} = -\sin\theta\left(-F^1_{13}\sin\theta + F^2_{13}\cos\theta\right)$$

$$\bar{T}^3_{13} = F^3_{13}\cos\theta \qquad\qquad\qquad\qquad \bar{T}^3_{23} = -F^3_{13}\sin\theta$$

$$\bar{T}^4_{14} = F^3_{13}\cos\theta + F^4_{24}\sin\theta \qquad\qquad \bar{T}^4_{24} = -F^3_{13}\sin\theta + F^4_{24}\cos\theta.$$

Thus the functions $\bar{T}^i_{jk}$ on Σ_1 satisfy the relations

$$\bar{T}^1_{13}\sin\theta + \bar{T}^1_{23}\cos\theta = \bar{T}^2_{13}\sin\theta + \bar{T}^2_{23}\cos\theta = \bar{T}^3_{13}\sin\theta + \bar{T}^3_{23}\cos\theta = 0,$$

$$\bar{T}^3_{13}\cos\theta - \bar{T}^3_{23}\sin\theta = \bar{T}^4_{14}\cos\theta - \bar{T}^4_{24}\sin\theta,$$

and are otherwise functionally independent in general.

Note that the dual framing $(V_0, V_1, V_2, V_3, V_4)$ to this adapted coframing is defined by the conditions that:

$$V_0 = -\frac{\partial}{\partial\theta}$$

$$(\pi_1)_* V_1 = (\cos\theta)\bar{V}_1 + (\sin\theta)\bar{V}_2$$

$$(\pi_1)_* V_2 = -(\sin\theta)\bar{V}_1 + (\cos\theta)\bar{V}_2$$

$$(\pi_1)_* V_3 = \bar{V}_3$$

$$(\pi_1)_* V_4 = \bar{V}_4,$$

together with the condition that the vector fields V_1, V_2, V_3, V_4 are tangent to the foliation whose leaves are the hypersurfaces on which θ is constant. Observe that:

- The rigid direction in D at each point in $\mathcal{X}$ is spanned by $(\pi_1)_*\left((\cos\theta)V_1 - (\sin\theta)V_2\right)$.
- These vector fields satisfy the bracket relations

$$[V_1, V_2] = -V_3$$

(3.3) $$\qquad [(\sin\theta)V_1 + (\cos\theta)V_2, V_3] = -V_4.$$

- The vector fields V_1, V_2, V_3, V_4 form a Lie subalgebra $\mathfrak{g}$ of the algebra of vector fields on Σ_1, and $[V_0, \mathfrak{g}] \subset \mathfrak{g}$.

3.2. Construction of the canonical sub-Finsler coframing on Σ. In this section we will use Cartan's method of equivalence [4] to construct a canonical coframing for the sub-Finsler structure Σ. Since the diagram

commutes, it is straightforward to verify that the null forms on Σ are spanned by $\rho^*(\omega^2)$, $\rho^*(\omega^3)$, and $\rho^*(\omega^4)$, that $\pi^{-1}D = \{\rho^*(\omega^3), \rho^*(\omega^4)\}^\perp$, and that $\rho^*(r\omega^1)$ is a monic form on Σ.

A local coframing $(\bar{\phi}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3, \bar{\eta}^4)$ on Σ will be called *0-adapted* if it has the following properties:

(1) $\bar{\eta}^1$ is a monic form,
(2) $\bar{\eta}^2, \bar{\eta}^3$, and $\bar{\eta}^4$ are null forms,
(3) $\pi^{-1}D = \{\bar{\eta}^3, \bar{\eta}^4\}^\perp$.

For example, the coframing

$$(3.4) \qquad \bar{\phi} = \rho^*(\alpha), \quad \bar{\eta}^1 = \rho^*(r\omega^1), \quad \bar{\eta}^2 = \rho^*(\omega^2), \quad \bar{\eta}^3 = \rho^*(\omega^3), \quad \bar{\eta}^4 = \rho^*(\omega^4)$$

is 0-adapted.

Any two 0-adapted coframings on Σ vary by a transformation of the form

$$(3.5) \qquad \begin{bmatrix} \tilde{\bar{\phi}} \\ \tilde{\bar{\eta}}^1 \\ \tilde{\bar{\eta}}^2 \\ \tilde{\bar{\eta}}^3 \\ \tilde{\bar{\eta}}^4 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_2 & b_3 & b_4 \\ 0 & 0 & c_2 & c_3 & c_4 \\ 0 & 0 & 0 & d_3 & d_4 \\ 0 & 0 & 0 & e_3 & e_4 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\phi} \\ \bar{\eta}^1 \\ \bar{\eta}^2 \\ \bar{\eta}^3 \\ \bar{\eta}^4 \end{bmatrix}$$

with $a_0 c_2 (d_3 e_4 - d_4 e_3) \neq 0$. The set of all 0-adapted coframings forms a principal fiber bundle $\mathcal{B}_0 \to \Sigma$, with structure group G_0 consisting of all matrices of the form (3.5). The right action of G_0 on sections $\sigma : \Sigma \to \mathcal{B}_0$ is given by $\sigma \cdot g = g^{-1}\sigma$. (This is the reason for the inverse matrix occurring in (3.5).)

There exist canonical 1-forms $\phi, \eta^1, \eta^2, \eta^3, \eta^4$ on $\mathcal{B}_0$ with the *reproducing property* that for any local section $\sigma : \Sigma \to \mathcal{B}_0$,

$$\sigma^*(\phi) = \bar{\phi}, \qquad \sigma^*(\eta^i) = \bar{\eta}^i.$$

These are referred to as the *semi-basic* forms on $\mathcal{B}_0$. A standard argument shows that there also exist (non-unique) 1-forms $\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i$ (referred to as *pseudo-connection forms* or, more succinctly, *connection forms*), linearly independent from the semi-basic forms, and functions T^i_{jk} on $\mathcal{B}_0$ (referred to as *torsion functions*) such that

$$(3.6) \qquad \begin{bmatrix} d\phi \\ d\eta^1 \\ d\eta^2 \\ d\eta^3 \\ d\eta^4 \end{bmatrix} = - \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & \beta_2 & \beta_3 & \beta_4 \\ 0 & 0 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & 0 & 0 & \delta_3 & \delta_4 \\ 0 & 0 & 0 & \epsilon_3 & \epsilon_4 \end{bmatrix} \wedge \begin{bmatrix} \phi \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \eta^4 \end{bmatrix} + \begin{bmatrix} 0 \\ T^1_{01}\,\phi \wedge \eta^1 \\ T^2_{01}\,\phi \wedge \eta^1 \\ T^3_{12}\,\eta^1 \wedge \eta^2 \\ T^4_{12}\,\eta^1 \wedge \eta^2 \end{bmatrix}.$$

(There are no torsion terms involving ϕ in $d\eta^3$, $d\eta^4$ because the system $\{\eta^3,\ \eta^4\}$ is well-defined on $\mathcal{X}$.) These are the *structure equations* of the G_0-structure $\mathcal{B}_0$. The semi-basic forms and connection forms together form a local coframing on $\mathcal{B}_0$.

We now proceed with the method of equivalence. As we make successive adaptations, we will keep the sub-Riemannian case – and particularly the bracket relations (3.3) – in mind, choosing normalizations so as to arrive at structure equations that are as similar to (3.2) as possible. We begin by examining how the functions T^i_{jk} vary if we change from one 0-adapted coframing to another. A straightforward computation shows that under a transformation of the form (3.5), we have

(3.7)
$$\tilde{T}^1_{01} = a_0 T^1_{01} - \frac{a_0 b_2}{c_2} T^2_{01}$$
$$\tilde{T}^2_{01} = \frac{a_0}{c_2} T^2_{01}$$
$$\tilde{T}^3_{12} = \frac{c_2}{(d_3 e_4 - e_3 d_4)}(e_4 T^3_{12} - d_4 T^4_{12})$$
$$\tilde{T}^4_{12} = \frac{c_2}{(d_3 e_4 - e_3 d_4)}(-e_3 T^3_{12} + d_3 T^4_{12}).$$

In particular, the function T^2_{01} is a *relative invariant*: if it vanishes for any 0-adapted coframing, then it vanishes for every 0-adapted coframing. Similarly, the *vector* $[T^3_{12},\ T^4_{12}]$ is a relative invariant. The coframing (3.4) has $T^2_{01} = r^{-1}$, $[T^3_{12},\ T^4_{12}] = [r^{-1},\ 0]$, so we can assume that these invariants are nonzero. (3.7) then implies that we can adapt coframings to arrange that
$$T^1_{01} = 0, \qquad T^2_{01} = 1, \qquad T^3_{12} = 1, \qquad T^4_{12} = 0.$$
A 0-adapted coframing satisfying these conditions will be called *1-adapted*. For example, if we set
$$dr = r_0\,\alpha + r_1\,\omega^1 + r_2\,\omega^2 + r_3\,\omega^3 + r_4\,\omega^4,$$
then the coframing

(3.8) $\bar{\phi} = \rho^*(r^{-1}\alpha)$, $\quad \bar{\eta}^1 = \rho^*(r\omega^1 - r_0\,\omega^2)$, $\quad \bar{\eta}^2 = \rho^*(\omega^2)$, $\quad \bar{\eta}^3 = \rho^*(r\omega^3)$, $\quad \bar{\eta}^4 = \rho^*(\omega^4)$

is 1-adapted. Any two 1-adapted coframings on Σ vary by a transformation of the form

(3.9)
$$\begin{bmatrix} \tilde{\phi} \\ \tilde{\eta}^1 \\ \tilde{\eta}^2 \\ \tilde{\eta}^3 \\ \tilde{\eta}^4 \end{bmatrix} = \begin{bmatrix} d_3 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & 0 & b_3 & b_4 \\ 0 & 0 & d_3 & c_3 & c_4 \\ 0 & 0 & 0 & d_3 & d_4 \\ 0 & 0 & 0 & 0 & e_4 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\phi} \\ \bar{\eta}^1 \\ \bar{\eta}^2 \\ \bar{\eta}^3 \\ \bar{\eta}^4 \end{bmatrix}$$

with $d_3 e_4 \neq 0$. The set of all 1-adapted coframings forms a principal fiber bundle $\mathcal{B}_1 \subset \mathcal{B}_0$, with structure group G_1 consisting of all matrices of the form (3.9). When restricted to $\mathcal{B}_1$, the connection forms β_2, ϵ_3, $\alpha_0 - \gamma_2$, $\delta_3 - \gamma_2$ become semi-basic, thereby introducing new

torsion terms into the structure equations of $\mathcal{B}_1$. Specifically, we compute:

$$0 = d(d\eta^1) \equiv \beta_2 \wedge \phi \wedge \eta^1 \mod \{\eta^2, \eta^3, \eta^4\}$$
$$\Rightarrow \beta_2 \equiv 0 \mod \{\phi, \eta^1, \eta^2, \eta^3, \eta^4\}.$$
$$0 = d(d\eta^2) \equiv (\gamma_2 - \alpha_0) \wedge \phi \wedge \eta^1 \mod \{\eta^2, \eta^3, \eta^4\}$$
$$\Rightarrow \alpha_0 \equiv \gamma_2 \mod \{\phi, \eta^1, \eta^2, \eta^3, \eta^4\}.$$
$$0 = d(d\eta^3) \equiv (\delta_3 - \gamma_2) \wedge \eta^1 \wedge \eta^2 \mod \{\eta^3, \eta^4\}$$
$$\Rightarrow \gamma_2 \equiv \delta_3 \mod \{\eta^1, \eta^2, \eta^3, \eta^4\}.$$
$$0 = d(d\eta^4) \equiv \epsilon_3 \wedge \eta^1 \wedge \eta^2 \mod \{\eta^3, \eta^4\}$$
$$\Rightarrow \epsilon_3 \equiv 0 \mod \{\eta^1, \eta^2, \eta^3, \eta^4\}.$$

By adding multiples of the semi-basic forms to the connection forms so as to absorb as much of the torsion as possible, we can arrange that the structure equations of $\mathcal{B}_1$ take the form

$$(3.10) \quad \begin{bmatrix} d\phi \\ d\eta^1 \\ d\eta^2 \\ d\eta^3 \\ d\eta^4 \end{bmatrix} = - \begin{bmatrix} \delta_3 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & \beta_3 & \beta_4 \\ 0 & 0 & \delta_3 & \gamma_3 & \gamma_4 \\ 0 & 0 & 0 & \delta_3 & \delta_4 \\ 0 & 0 & 0 & 0 & \epsilon_4 \end{bmatrix} \wedge \begin{bmatrix} \phi \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \eta^4 \end{bmatrix} + \begin{bmatrix} 0 \\ T^1_{02}\,\phi \wedge \eta^2 + T^1_{12}\,\eta^1 \wedge \eta^2 \\ \phi \wedge \eta^1 + T^2_{12}\,\eta^1 \wedge \eta^2 \\ \eta^1 \wedge \eta^2 \\ T^4_{13}\,\eta^1 \wedge \eta^3 + T^4_{23}\,\eta^2 \wedge \eta^3 \end{bmatrix}.$$

We now repeat this process. Under a transformation of the form (3.9), we have

$$(3.11) \quad \begin{aligned} \tilde{T}^1_{02} &= d_3^2 T^1_{02} \\ \tilde{T}^1_{12} &= d_3 T^1_{12} + a_1 d_3 T^1_{02} - b_3 \\ \tilde{T}^2_{12} &= T^2_{12} + \frac{d_4}{e_4} T^4_{13} - \frac{(a_2 + 2c_3)}{d_3} \\ \tilde{T}^4_{13} &= \frac{d_3}{e_4} T^4_{13} \\ \tilde{T}^4_{23} &= \frac{d_3^2}{e_4} T^4_{23} \end{aligned}$$

Observe that:

- T^1_{02} is a relative invariant which transforms by a square, so its sign is fixed. The coframing (3.8) is 1-adapted, and if we set

$$dr_0 = r_{00}\,\alpha + r_{01}\,\omega^1 + r_{02}\,\omega^2 + r_{03}\,\omega^3 + r_{04}\,\omega^4,$$

it has $T^1_{02} = -r(r + r_{00})$. The condition that each fiber of Σ be a strongly convex curve enclosing the origin is exactly the condition that this quantity be negative, so we can assume that $T^1_{02} < 0$.

- T^4_{13}, T^4_{23} are also relative invariants. The coframing (3.8) has

$$T^4_{13} = \frac{\sin\theta}{r^2}, \qquad T^4_{23} = \frac{(r\cos\theta + r_0 \sin\theta)}{r^2};$$

thus we see that T^4_{13} and T^4_{13} do not vanish simultaneously.

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With these observations, equation (3.11) implies that we can adapt coframings to arrange that $T^1_{02} = -1$, and after doing so, the quantity $(T^4_{13})^2 + (T^4_{23})^2$ becomes a nonvanishing relative invariant. So we can further adapt coframings to arrange that

$$T^1_{02} = -1, \qquad (T^4_{13})^2 + (T^4_{23})^2 = 1.$$

A 1-adapted coframing satisfying these conditions will be called *2-adapted*. (Note that in principle, we could also adapt so that $T^1_{12} = 0$. However, this would make it harder to write down an explicit 2-adapted coframing, which will be convenient in order to make some observations; hence we will postpone this normalization until the next round of adaptations.) For example, the coframing

$$(3.12) \qquad \bar\phi = \rho^* \left(\frac{\sqrt{r + r_{00}}}{\sqrt{r}} \alpha \right), \quad \bar\eta^1 = \rho^* \left(r\,\omega^1 - r_0\,\omega^2 \right), \quad \bar\eta^2 = \rho^* \left(\sqrt{r(r + r_{00})}\,\omega^2 \right),$$

$$\bar\eta^3 = \rho^* \left(r^{3/2}\sqrt{r + r_{00}}\,\omega^3 \right), \quad \bar\eta^4 = \rho^* \left(\frac{r^3(r + r_{00})}{\sqrt{(r\cos\theta + r_0\sin\theta)^2 + r(r + r_{00})\sin^2\theta}}\,\omega^4 \right)$$

is 2-adapted. Any two 2-adapted coframings on Σ vary by a transformation of the form

$$(3.13) \qquad \begin{bmatrix} \tilde\phi \\ \tilde\eta^1 \\ \tilde\eta^2 \\ \tilde\eta^3 \\ \tilde\eta^4 \end{bmatrix} = \begin{bmatrix} s_1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & 0 & b_3 & b_4 \\ 0 & 0 & s_1 & c_3 & c_4 \\ 0 & 0 & 0 & s_1 & d_4 \\ 0 & 0 & 0 & 0 & s_2 \end{bmatrix}^{-1} \begin{bmatrix} \bar\phi \\ \bar\eta^1 \\ \bar\eta^2 \\ \bar\eta^3 \\ \bar\eta^4 \end{bmatrix}$$

with $s_1, s_2 = \pm 1$. The set of all 2-adapted coframings forms a principal fiber bundle $\mathcal{B}_2 \subset \mathcal{B}_1$, with structure group G_2 consisting of all matrices of the form (3.13). When restricted to $\mathcal{B}_2$, the connection forms δ_3, ϵ_4 become semi-basic, thereby introducing new torsion terms into the structure equations of $\mathcal{B}_2$. Specifically, we compute:

$$0 = d(d\eta^4) \equiv [dT^4_{13} - T^4_{13}(\delta_3 - \epsilon_4) + T^4_{23}\phi] \wedge \eta^1 \wedge \eta^3 \mod \{\eta^2,\, \eta^4\}$$
$$\Rightarrow dT^4_{13} \equiv T^4_{13}(\delta_3 - \epsilon_4) - T^4_{23}\phi \mod \{\eta^1,\, \eta^2,\, \eta^3,\, \eta^4\},$$
$$0 = d(d\eta^4) \equiv [dT^4_{23} - T^4_{23}(2\delta_3 - \epsilon_4) - T^4_{13}\phi] \wedge \eta^2 \wedge \eta^3 \mod \{\eta^1,\, \eta^4\}$$
$$\Rightarrow dT^4_{23} \equiv T^4_{23}(2\delta_3 - \epsilon_4) + T^4_{13}\phi \mod \{\eta^1,\, \eta^2,\, \eta^3,\, \eta^4\},$$

and therefore,

$$0 = d[(T^4_{13})^2 + (T^4_{23})^2] \equiv [2 + 2(T^4_{23})^2]\delta_3 - 2\epsilon_4 \mod \{\eta^1,\, \eta^2,\, \eta^3,\, \eta^4\}.$$

In addition, we have

$$0 = d(d\eta^1) \equiv 2\delta_3 \wedge \phi \wedge \eta^2 \mod \{\eta^1,\, \eta^3,\, \eta^4\}$$
$$\Rightarrow \delta_3 \equiv 0 \mod \{\phi,\, \eta^1,\, \eta^2,\, \eta^3,\, \eta^4\}.$$

Observe that under a transformation of the form (3.13), we have

$$(3.14) \qquad \tilde{T}^4_{13} = s_1 s_2 T^4_{13}, \qquad \tilde{T}^4_{23} = s_2 T^4_{23}.$$

Therefore, the torsion functions T^4_{13}, T^4_{23} on $\mathcal{B}_2$ descend to well-defined functions on Σ up to sign, and there exists a function Θ on Σ, well-defined up to sign and modulo π, with the property that

$$T^4_{13} = \sin\Theta, \qquad T^4_{23} = \cos\Theta.$$

We can eliminate the ambiguity in the function Θ as follows: let $H_2 \subset G_2$ denote the identity component. The quotient $\tilde{\Sigma} = \mathcal{B}_2/H_2$ is a principal bundle over Σ with discrete fibers isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and Θ is a well-defined function on $\tilde{\Sigma}$. Henceforth, we will regard the bundle $\mathcal{B}_2$ as an H_2-structure over $\tilde{\Sigma}$; this will eliminate the sign ambiguities in the group G_2 and simplify the remainder of our computations.

By adding multiples of the semi-basic forms to the connection forms so as to absorb as much of the torsion as possible, we can arrange that the structure equations of $\mathcal{B}_2$ take the form

$$(3.15) \quad
\begin{bmatrix} d\phi \\ d\eta^1 \\ d\eta^2 \\ d\eta^3 \\ d\eta^4 \end{bmatrix}
= -
\begin{bmatrix}
0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
0 & 0 & 0 & \beta_3 & \beta_4 \\
0 & 0 & 0 & \gamma_3 & \gamma_4 \\
0 & 0 & 0 & 0 & \delta_4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\wedge
\begin{bmatrix} \phi \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \eta^4 \end{bmatrix}$$

$$+
\begin{bmatrix}
0 \\
-\phi \wedge \eta^2 + T^1_{12}\,\eta^1 \wedge \eta^2 \\
\phi \wedge \eta^1 + T^2_{02}\,\phi \wedge \eta^2 + T^2_{12}\,\eta^1 \wedge \eta^2 \\
\eta^1 \wedge \eta^2 + \left[T^2_{02}\,\phi + T^3_{13}\,\eta^1 + T^3_{23}\,\eta^2 \right] \wedge \eta^3 \\
\left[(\sin\Theta)\,\eta^1 + (\cos\Theta)\,\eta^2 \right] \wedge \eta^3 + \left[T^2_{02}(1 + \cos^2\Theta)\,\phi + T^4_{14}\,\eta^1 + T^4_{24}\,\eta^2 + T^4_{34}\,\eta^3 \right] \wedge \eta^4
\end{bmatrix}.$$

Let us pause for a moment and compare equations (3.15) to (3.2). First, observe that under a transformation of the form (3.13), we have

$$\tilde{T}^2_{02} = T^2_{02};$$

i.e., T^2_{02} is now a well-defined function on $\tilde{\Sigma}$. The following proposition shows how it may be interpreted as the Cartan scalar of the sub-Finsler structure:

Proposition 3.1. $T^2_{02} \equiv 0$ *if and only if the sub-Finsler structure is sub-Riemannian.*

Proof. It is straightforward to compute that for the coframing (3.12),

$$T^2_{02} = \rho^* \left(\frac{3r_0 r_{00} + 4 r r_0 + r r_{000}}{2\sqrt{r}(r + r_{00})^{3/2}} \right),$$

with r_{000} defined by

$$dr_{00} = r_{000}\,\alpha + r_{001}\,\omega^1 + r_{002}\,\omega^2 + r_{003}\,\omega^3 + r_{004}\,\omega^4.$$

Note that r is a function on Σ_1, and that $r_0 = -\frac{\partial r}{\partial \theta}$, $r_{00} = \frac{\partial^2 r}{\partial \theta^2}$, $r_{000} = -\frac{\partial^3 r}{\partial \theta^3}$. For simplicity, fix any point $x \in X$ and restrict to the fiber Σ_x, with θ as a local coordinate. Then

$$T_{02}^2 = 0 \iff 3r_0 r_{00} + 4rr_0 + rr_{000} = 0$$
$$\iff \frac{d}{d\theta}(rr_{00} + r_0^2 + 2r^2) = 0$$
$$\iff rr_{00} + r_0^2 + 2r^2 = c_1$$
$$\iff (\tfrac{1}{2}r^2)_{00} + 2r^2 = c_1$$
$$\iff r^2 = A\cos^2(\theta - C) + B\sin^2(\theta - C)$$

for some constants A, B, C. This last equation holds precisely when the function r^{-1} is the radial function (in polar coordinates) for an ellipse centered at the origin, which is true for all x if and only if the sub-Finsler structure is sub-Riemannian. $\qquad\square$

Henceforth, we will denote T_{02}^2 by I, in keeping with the usual notation for the Cartan scalar.

Next, note that the function Θ on the sub-Finsler structure $\tilde{\Sigma}$ plays the role of the angle function θ on the sub-Riemannian structure Σ_1. Θ has the following geometric interpretation: at each point $\mathbf{u} \in \tilde{\Sigma}$, consider the 2-plane $\pi^{-1}(D) \subset T_\mathbf{u}\tilde{\Sigma}$. By a well-known construction in Finsler geometry, the sub-Finsler metric on D induces a Riemannian metric on $\pi^{-1}(D)$. Let $V_1, V_2 \in \pi^{-1}(D)$ be vectors such that:

- $\pi_*(V_1) \in D$ is a rigid direction, and
- $\pi_*(V_2) = \mathbf{u} \in D$.

Then Θ is simply the angle between V_1 and V_2 in the induced Riemannian metric on $\pi^{-1}(D)$.

We now consider how Θ varies on the fibers of the projection $\pi : \tilde{\Sigma} \to \mathcal{X}$: computing $d(d\eta^4) \equiv 0 \mod \eta^1, \eta^4$ and $d(d\eta^4) \equiv 0 \mod \eta^2, \eta^4$ shows that

$$d\Theta \equiv -(1 - I\sin\Theta\cos\Theta)\,\phi \mod \eta^1, \eta^2, \eta^3, \eta^4.$$

(Note that in the sub-Riemannian case, we have $\phi = \alpha = -d\theta$ and $I = 0$, so this is consistent with our observation that $\Theta = \theta$ in that case.)

Motivated by the sub-Riemannian case, we would like to adapt coframes to arrange that ϕ is an exact multiple of $d\Theta$ and hence an integrable 1-form. Unfortunately, this is not always possible: one can construct examples for which Θ is not a monotonic function of θ and the function $1 - I\sin\Theta\cos\Theta$ vanishes on a nonempty subset of $\tilde{\Sigma}$. In order to avoid this problem, we will restrict to sub-Finsler structures with the property that $|I| < 2$.

Definition 3.2. *A sub-Finsler structure Σ is called* tame *if its Cartan scalar I satisfies $|I| < 2$ at each point of Σ.*

Henceforth, we shall consider only tame sub-Finsler structures.

Observe that under a transformation of the form (3.13), we have

(3.16)
$$\tilde{T}_{12}^1 = T_{12}^1 - a_1 - b_3$$
$$\tilde{T}_{12}^2 = T_{12}^2 + a_1 I - a_2 - c_3$$
$$\tilde{T}_{13}^3 = T_{13}^3 + a_1 I + c_3 - d_4 \sin\Theta$$
$$\tilde{T}_{23}^3 = T_{23}^3 + a_2 I - b_3 - d_4 \cos\Theta.$$

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It follows that

$$\tilde{T}^3_{13} \sin \Theta + \tilde{T}^3_{23} \cos \Theta = T^3_{13} \sin \Theta + T^3_{23} \cos \Theta$$
$$+ (a_1 \sin \Theta + a_2 \cos \Theta)I + c_3 \sin \Theta - b_3 \cos \Theta - d_4.$$

Therefore, we can adapt coframings to arrange that

$$d\Theta = -(1 - I \sin \Theta \cos \Theta)\,\phi$$

(note that this condition uniquely determines ϕ) and that

$$T^1_{12} = T^2_{12} = T^3_{13} \sin \Theta + T^3_{23} \cos \Theta = 0.$$

A coframing satisfying these conditions will be called *3-adapted*. For such a coframing, we will write

$$T^3_{13} = B_3 \cos \Theta, \qquad T^3_{23} = -B_3 \sin \Theta.$$

Any two 3-adapted coframings on $\tilde{\Sigma}$ vary by a transformation of the form

(3.17)
$$
\begin{bmatrix} \tilde{\phi} \\ \tilde{\eta}^1 \\ \tilde{\eta}^2 \\ \tilde{\eta}^3 \\ \tilde{\eta}^4 \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & b_4 \\
0 & 0 & 1 & 0 & c_4 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix} \bar{\phi} \\ \bar{\eta}^1 \\ \bar{\eta}^2 \\ \bar{\eta}^3 \\ \bar{\eta}^4 \end{bmatrix}.
$$

The set of all 3-adapted coframings forms a principal fiber bundle $\mathcal{B}_3 \subset \mathcal{B}_2$, with structure group G_3 consisting of all matrices of the form (3.17). When restricted to $\mathcal{B}_3$, the connection forms $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_3, \gamma_3, \delta_4$ become semi-basic, thereby introducing new torsion terms into the structure equations of $\mathcal{B}_3$. By adding multiples of the semi-basic forms to the connection forms so as to absorb as much of the torsion as possible (and recalling that ϕ is now integrable), we can arrange that the structure equations of $\mathcal{B}_3$ take the form

(3.18)
$$
\begin{bmatrix} d\phi \\ d\eta^1 \\ d\eta^2 \\ d\eta^3 \\ d\eta^4 \end{bmatrix}
= -
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_4 \\
0 & 0 & 0 & 0 & \gamma_4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\wedge
\begin{bmatrix} \phi \\ \eta^1 \\ \eta^2 \\ \eta^3 \\ \eta^4 \end{bmatrix}
+
$$

$$
\begin{bmatrix}
\phi \wedge (T^0_{01}\eta^1 + T^0_{02}\eta^2 + T^0_{03}\eta^3 + T^0_{04}\eta^4) \\
-\phi \wedge \eta^2 + (T^1_{03}\phi + T^1_{13}\eta^1 + T^1_{23}\eta^2) \wedge \eta^3 \\
\phi \wedge \eta^1 + I\phi \wedge \eta^2 + (T^2_{03}\phi + T^2_{13}\eta^1 + T^2_{23}\eta^2) \wedge \eta^3 \\
\eta^1 \wedge \eta^2 + \left[I\phi + (B_3 \cos \Theta)\eta^1 - (B_3 \sin \Theta)\eta^2 \right] \wedge \eta^3 + (T^3_{04}\phi + T^3_{14}\eta^1 + T^3_{24}\eta^2 + T^3_{34}\eta^3) \wedge \eta^4 \\
\left[(\sin \Theta)\eta^1 + (\cos \Theta)\eta^2 \right] \wedge \eta^3 + \left[(1 + \cos^2 \Theta)I\phi + T^4_{14}\eta^1 + T^4_{24}\eta^2 + T^4_{34}\eta^3 \right] \wedge \eta^4
\end{bmatrix}.
$$

Computing $d(d\eta^1) \equiv 0 \mod \eta^3, \eta^4$ shows that

$$T^0_{01} = -T^1_{03}.$$

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Under a transformation of the form (3.17), we have

$$\tilde{T}^1_{13} = T^1_{13} - b_4 \sin \Theta$$

(3.19)
$$\tilde{T}^1_{23} = T^1_{23} - b_4 \cos \Theta$$
$$\tilde{T}^2_{13} = T^2_{13} - c_4 \sin \Theta$$
$$\tilde{T}^2_{23} = T^2_{23} - c_4 \cos \Theta.$$

It follows that

$$\tilde{T}^1_{13} \sin \Theta + \tilde{T}^1_{23} \cos \Theta = T^1_{13} \sin \Theta + T^1_{23} \cos \Theta - b_4$$
$$\tilde{T}^2_{13} \sin \Theta + \tilde{T}^2_{23} \cos \Theta = T^2_{13} \sin \Theta + T^2_{23} \cos \Theta - c_4.$$

Therefore, we can adapt coframings to arrange that

$$T^1_{13} \sin \Theta + T^1_{23} \cos \Theta = T^2_{13} \sin \Theta + T^2_{23} \cos \Theta = 0.$$

A coframing satisfying these conditions will be called *4-adapted*. For such a coframing, we will write

$$T^1_{13} = B_1 \cos \Theta \qquad T^1_{23} = -B_1 \sin \Theta$$
$$T^2_{13} = B_2 \cos \Theta \qquad T^2_{23} = -B_2 \sin \Theta.$$

The set of 4-adapted coframings is an (e)-structure $\mathcal{B}_4 \subset \mathcal{B}_3$ on $\tilde{\Sigma}$. In other words, there is a unique 4-adapted coframing on $\tilde{\Sigma}$, and the set of all 4-adapted coframings on Σ forms a principal fiber bundle $\mathcal{B}_4 \to \Sigma$, with discrete structure group $G_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. When restricted to $\mathcal{B}_4$, the connection forms β_4, γ_4 become semi-basic, and the structure equations for a 4-adapted coframing take the form

$$d\phi = \phi \wedge \left(-T^1_{03}\, \eta^1 + T^0_{02}\, \eta^2 + T^0_{03}\, \eta^3 + T^0_{04}\, \eta^4 \right)$$

$$d\eta^1 = -\phi \wedge \eta^2 + \left[T^1_{03}\, \phi + (B_1 \cos \Theta)\, \eta^1 - (B_1 \sin \Theta)\, \eta^2 \right] \wedge \eta^3$$
$$+ \left(T^1_{04}\, \phi + T^1_{14}\, \eta^1 + T^1_{24}\, \eta^2 + T^1_{34}\, \eta^3 \right) \wedge \eta^4$$

(3.20)
$$d\eta^2 = \phi \wedge \eta^1 + I\, \phi \wedge \eta^2 + \left[T^2_{03}\, \phi + (B_2 \cos \Theta)\, \eta^1 - (B_2 \sin \Theta)\, \eta^2 \right] \wedge \eta^3$$
$$+ \left(T^2_{04}\, \phi + T^2_{14}\, \eta^1 + T^2_{24}\, \eta^2 + T^2_{34}\, \eta^3 \right) \wedge \eta^4$$

$$d\eta^3 = \eta^1 \wedge \eta^2 + \left[I\, \phi + (B_3 \cos \Theta)\, \eta^1 - (B_3 \sin \Theta)\, \eta^2 \right] \wedge \eta^3$$
$$+ \left(T^3_{04}\, \phi + T^3_{14}\, \eta^1 + T^3_{24}\, \eta^2 + T^3_{34}\, \eta^3 \right) \wedge \eta^4$$

$$d\eta^4 = \left[(\sin \Theta)\, \eta^1 + (\cos \Theta)\, \eta^2 \right] \wedge \eta^3$$
$$+ \left[(1 + \cos^2 \Theta) I\, \phi + T^4_{14}\, \eta^1 + T^4_{24}\, \eta^2 + T^4_{34}\, \eta^3 \right] \wedge \eta^4.$$

Differentiating these equations yields some relations among the torsion coefficients. First we compute:

$$0 = d(d\eta^4) \equiv (B_3 - T^4_{14} \cos \Theta + T^4_{24} \sin \Theta)\, \eta^1 \wedge \eta^2 \wedge \eta^3 \quad \mathrm{mod}\ \phi, \eta^4.$$

Therefore, we can write

$$T^4_{14} = B_4 \sin \Theta + B_3 \cos \Theta, \qquad T^4_{24} = B_4 \cos \Theta - B_3 \sin \Theta$$

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for some function B_4 on $\mathcal{B}_4$. Next, we have:

$$0 = d(d\eta^2) \equiv \left[dI + (T^0_{02} + T^2_{03} + IT^1_{03})\,\eta^1\right] \wedge \phi \wedge \eta^2 \quad \bmod \eta^3, \eta^4.$$

Therefore,

$$dI \equiv -(T^0_{02} + T^2_{03} + IT^1_{03})\,\eta^1 \quad \bmod \phi, \eta^2, \eta^3, \eta^4.$$

Using this result, we compute

$$0 = d(d\Theta) \equiv \left[T^1_{03} + (T^0_{02} + T^2_{03})\sin\Theta\cos\Theta\right] \quad \bmod \eta^2, \eta^3, \eta^4.$$

This yields the relation

(3.21) $$T^1_{03} + (T^0_{02} + T^2_{03})\sin\Theta\cos\Theta = 0.$$

The final relation requires a bit more effort. We need to introduce the following notation for the derivatives of I:

$$dI = I_0\,\phi - (T^0_{02} + T^2_{03} + IT^1_{03})\,\eta^1 + I_2\,\eta^2 + I_3\,\eta^3 + I_4\,\eta^4.$$

First we compute:

$$0 = d(d\eta^3)$$
$$\equiv \left[(\cos\Theta)dB_3 + (T^0_{02} + 2T^2_{03} - T^3_{04}\sin\Theta - IB_3\sin^2\Theta\cos\Theta)\,\phi\right] \wedge \eta^1 \wedge \eta^3$$
$$\bmod \eta^2, \eta^4$$

$$0 = d(d\eta^3)$$
$$\equiv \left[(-\sin\Theta)dB_3 - (T^1_{03} - IT^0_{02} + I_2 + T^3_{04}\cos\Theta + IB_3\sin\Theta(\cos^2\Theta + 1))\,\phi\right] \wedge \eta^2 \wedge \eta^3$$
$$\bmod \eta^1, \eta^4.$$

Therefore, modulo $\{\eta^1, \eta^2, \eta^3, \eta^4\}$, we have:

(3.22) $$(\cos\Theta)dB_3 \equiv -(T^0_{02} + 2T^2_{03} - T^3_{04}\sin\Theta - IB_3\sin^2\Theta\cos\Theta)\,\phi$$

(3.23) $$(\sin\Theta)dB_3 \equiv (IT^0_{02} - T^1_{03} - I_2 - T^3_{04}\cos\Theta - IB_3\sin\Theta(\cos^2\Theta + 1))\,\phi.$$

Computing $(\cos\Theta)(3.22) + (\sin\Theta)(3.23)$ yields:

$$dB_3 \equiv \left[-IB_3\sin^2\Theta + (IT^2_{02} - T^1_{03} - I_2)\sin\Theta - (T^0_{02} + 2T^2_{03})\cos\Theta\right]\phi \quad \bmod \eta^1, \eta^2, \eta^3, \eta^4,$$

and computing $(\sin\Theta)(3.22) - (\cos\Theta)(3.23)$ modulo $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ yields:

$$I_2\cos\Theta = -T^3_{04} + (T^0_{02} + 2T^2_{03})\sin\Theta + (IT^0_{02} - T^1_{03})\cos\Theta - 2IB_3\sin\Theta\cos\Theta.$$

Now we compute:

$$0 = d(d\eta^4)$$
$$\equiv \left[(\sin\Theta)dB_4 + (T_{02}^0\cos^2\Theta - T_{03}^2\sin^2\Theta + 2T_{04}^3\sin\Theta + IB_4\sin\Theta\cos^2\Theta)\,\phi\right] \wedge \eta^1 \wedge \eta^4$$
$$\mathrm{mod}\ \eta^2, \eta^3$$

$$0 = d(d\eta^4)$$
$$\equiv \left[(\cos\Theta)dB_4 + \left(IB_4\cos^3\Theta + 2IB_3\sin\Theta\cos^2\Theta - T_{03}^1\sin^2\Theta\right.\right.$$
$$\left.\left. - (2T_{03}^2 + T_{02}^0)\sin\Theta\cos\Theta + 3T_{04}^3\cos\Theta\right)\phi\right] \wedge \eta^2 \wedge \eta^4$$
$$\mathrm{mod}\ \eta^1, \eta^3.$$

Therefore, modulo $\{\eta^1, \eta^2, \eta^3, \eta^4\}$, we have:

(3.24) $\qquad (\sin\Theta)dB_4 = -(T_{02}^0\cos^2\Theta - T_{03}^2\sin^2\Theta + 2T_{04}^3\sin\Theta + IB_4\sin\Theta\cos^2\Theta)\,\phi$

(3.25) $\qquad (\cos\Theta)dB_4 = -\left(IB_4\cos^3\Theta + 2IB_3\sin\Theta\cos^2\Theta - T_{03}^1\sin^2\Theta\right.$
$$\left. - (2T_{03}^2 + T_{02}^0)\sin\Theta\cos\Theta + 3T_{04}^3\cos\Theta\right)\phi.$$

Computing $(\cos\Theta)$ times (3.24) $-(\sin\Theta)$ times (3.25) yields the final relation:

(3.26) $\ -2IB_3\sin^2\Theta\cos^2\Theta - T_{03}^1\sin^3\Theta + T_{03}^2\sin^2\Theta\cos\Theta - T_{04}^3\sin\Theta\cos\Theta + T_{02}^0\cos\Theta = 0.$

If we solve equation (3.21) for T_{03}^1 and substitute into (3.26), it becomes

$$T_{02}^0(2 - \cos^4\Theta) + T_{03}^2\sin^2\Theta(2 + \cos^2\Theta) - 2IB_3\sin^2\Theta\cos\Theta - T_{04}^3\sin\Theta = 0.$$

Further differentiation yields only relations among the derivatives of the T_{jk}^i.

We summarize this discussion as:

Theorem 3.3. *Let Σ be a tame sub-Finsler structure on an Engel distribution $(\mathcal{X}, D)$. Then there exists a well-defined principal fiber bundle $\mathcal{B}_4 \to \Sigma$, with fiber group $G_4 = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, and a canonical coframing $(\phi, \eta^1, \eta^2, \eta^3, \eta^4)$ on $\mathcal{B}_4$ whose structure equations have the form*

$$d\phi = \phi \wedge \left[(T_{02}^0 + T_{03}^2)\sin\Theta\cos\Theta\,\eta^1 + T_{02}^0\,\eta^2 + T_{03}^0\,\eta^3 + T_{04}^0\,\eta^4\right]$$

$$d\eta^1 = -\phi \wedge \eta^2 + \left[-(T_{02}^0 + T_{03}^2)\sin\Theta\cos\Theta\,\phi + (B_1\cos\Theta)\,\eta^1 - (B_1\sin\Theta)\,\eta^2\right] \wedge \eta^3$$
$$+ \left(T_{04}^1\,\phi + T_{14}^1\,\eta^1 + T_{24}^1\,\eta^2 + T_{34}^1\,\eta^3\right) \wedge \eta^4$$

(3.27) $\quad d\eta^2 = \phi \wedge \eta^1 + I\,\phi \wedge \eta^2 + \left[T_{03}^2\,\phi + (B_2\cos\Theta)\,\eta^1 - (B_2\sin\Theta)\,\eta^2\right] \wedge \eta^3$
$$+ \left(T_{04}^2\,\phi + T_{14}^2\,\eta^1 + T_{24}^2\,\eta^2 + T_{34}^2\,\eta^3\right) \wedge \eta^4$$

$$d\eta^3 = \eta^1 \wedge \eta^2 + \left[I\,\phi + (B_3\cos\Theta)\,\eta^1 - (B_3\sin\Theta)\,\eta^2\right] \wedge \eta^3$$
$$+ \left(T_{04}^3\,\phi + T_{14}^3\,\eta^1 + T_{24}^3\,\eta^2 + T_{34}^3\,\eta^3\right) \wedge \eta^4$$

$$d\eta^4 = \left[(\sin\Theta)\,\eta^1 + (\cos\Theta)\,\eta^2\right] \wedge \eta^3$$
$$+ \left[(1 + \cos^2\Theta)I\,\phi + T_{14}^4\,\eta^1 + T_{24}^4\,\eta^2 + T_{34}^4\,\eta^3\right] \wedge \eta^4$$

for some functions T^i_{jk} on $\mathcal{B}_4$. These functions satisfy the relation

(3.28) $T^0_{02}(2 - \cos^4 \Theta) + T^2_{03} \sin^2 \Theta (2 + \cos^2 \Theta) - 2IB_3 \sin^2 \Theta \cos \Theta - T^3_{04} \sin \Theta = 0.$

The dual framing $(V_0, V_1, V_2, V_3, V_4)$ to this coframing satisfies conditions analogous to those in the sub-Riemannian case, except for the last one:

- The rigid direction in D at each point in $\mathfrak{X}$ is spanned by $\pi_* ((\cos \Theta)V_1 - (\sin \Theta)V_2)$.
- $[V_1, V_2] = -V_3$.
- $[(\sin \Theta)V_1 + (\cos \Theta)V_2, V_3] = -V_4$.
- Σ has a natural foliation by hypersurfaces of the form $\Theta = \Theta_0$. The vector fields V_1, V_2, V_3, V_4 are tangent to this foliation and form a Lie subalgebra $\mathfrak{g}$ of the algebra of vector fields on Σ. However, it is no longer true that $[V_0, \mathfrak{g}] \subset \mathfrak{g}$.

4. Geodesic equations

In this section we consider the problem of finding geodesics of the sub-Finsler structure. Recall that the sub-Finsler length of a horizontal curve $\gamma : [a, b] \to \mathfrak{X}$ is given by

$$(4.1) \qquad \mathcal{L}(\gamma) = \int_a^b F(\gamma'(t)) \, dt.$$

Finding critical points of this functional amounts to solving a constrained variational problem. However, the existence of rigid curves presents a particular challenge, as these curves have no C^1-variations, and thus traditional methods in the calculus of variations cannot be applied. Thus these curves must be considered as a separate case. In the remainder of this section, we will use the variational methods described by Griffiths in [5] to compute the geodesic equations for *regular* geodesics; we will consider the geodesic problem for rigid curves separately in §5.

Since the geodesic equations are local, we may work in an orientable neighborhood of $\mathfrak{X}$. So choose orientations for the rigid line field on $\mathfrak{X}$ and for the distribution D, and consider the unique coframing $(\phi, \eta^1, \eta^2, \eta^3, \eta^4)$ on Σ which is compatible with these choices of orientation.

Every horizontal curve $\gamma : [a, b] \to \mathfrak{X}$ has a unique lift to an integral curve $\hat{\gamma} : [a, b] \to \Sigma$ of the differential system $\hat{\mathcal{I}} = \{\eta^2, \eta^3, \eta^4\}$ with $\eta^1(\hat{\gamma}'(t)) \neq 0$ and $\hat{\gamma}(t)$ a positive multiple of the vector $\gamma'(t)$. The sub-Finsler length of γ is then equal to the integral of the monic 1-form η^1 along the lifted curve $\hat{\gamma}$. The problem of finding critical curves of the sub-Finsler length functional among horizontal curves is thus equivalent to finding critical curves of

$$(4.2) \qquad \hat{\mathcal{L}}(\hat{\gamma}) = \int_{\hat{\gamma}} \eta^1$$

among integral curves $\hat{\gamma}$ of $\hat{\mathcal{I}} = \{\eta^2, \eta^3, \eta^4\}$ on Σ.

Theorem 4.1. *The critical curves of $\hat{\mathcal{L}}$ among integral curves of $\hat{\mathcal{I}}$ on Σ are precisely the projections of integral curves, with transversality condition $\eta^1 \neq 0$, of the differential system*

$\mathcal{J} = \{\eta^2, \eta^3, \eta^4, \phi - \lambda \eta^1,$

$\qquad d\lambda - \left(-B_1 \cos \Theta + ((T^0_{02} + T^2_{03}) \cos \Theta \sin \Theta - B_3 \cos \Theta)\lambda - (\sin \Theta)\mu - I\lambda^2\right) \eta^1,$

$\qquad d\mu - \left(-T^1_{14} - (T^1_{04} + T^3_{14})\lambda - (B_3 \cos \Theta + B_4 \sin \Theta)\mu - T^3_{04}\lambda^2 - (1 + \cos^2 \Theta)I\lambda\mu\right) \eta^1\}$

on $\mathcal{Y} \cong \mathcal{B}_4 \times \mathbb{R}^2$, where λ, μ are the coordinates on the $\mathbb{R}^2$ fiber.

Proof. Within our orientable neighborhood, Σ can be identified with one sheet of the four-sheeted cover $\mathcal{B}_4 \to \Sigma$, corresponding to the choice of orientations for the coframing $(\phi, \eta^1, \eta^2, \eta^3, \eta^4)$. (For simplicity we will continue to use the notation $\mathcal{B}_4$ for this set of coframes.) Thus we can regard $\hat{\gamma}$ as a curve in $\mathcal{B}_4$; this corresponds to choosing a 4-adapted coframing along the horizontal curve γ so that the vector V_1 dual to η^1 points in the direction of the velocity vector of the curve.

Following the algorithm in [5], we define a submanifold $\mathcal{Z} \subset T^*\mathcal{B}_4$ as follows: for each $x \in \mathcal{B}_4$, let $\mathcal{Z}_x = \eta^1(x) + \text{span}\{\hat{\mathcal{I}}_x\}$ and let

$$
(4.3) \qquad \mathcal{Z} = \bigcup_{x \in \mathcal{B}_4} \mathcal{Z}_x.
$$

Let ζ be the pullback to $\mathcal{Z}$ of the canonical 1-form on $T^*\mathcal{B}_4$. By the "self-reproducing" property of ζ, we may write

$$
(4.4) \qquad \zeta = \eta^1 + \lambda_2\eta^2 + \lambda_3\eta^3 + \lambda_4\eta^4
$$

(where we have suppressed the obvious pullbacks in our notation). According to the general theory described in [5], the critical points of the functional

$$
(4.5) \qquad \tilde{\mathcal{L}}(\tilde{\gamma}) = \int_{\tilde{\gamma}} \zeta
$$

among unconstrained curves $\tilde{\gamma}$ on $\mathcal{Z}$ project to critical curves of $\hat{\mathcal{L}}$ among integral curves $\hat{\gamma}$ of $\hat{\mathcal{I}}$ on $\mathcal{B}_4$; moreover, a curve $\tilde{\gamma}$ on $\mathcal{Z}$ is a critical curve of $\tilde{\mathcal{L}}$ if and only if $\tilde{\gamma}'(t) \lrcorner d\zeta|_{\tilde{\gamma}(t)} = 0$.

A straightforward computation shows that

$$
\begin{aligned}
d\zeta = \; & d\lambda_2 \wedge \eta^2 + d\lambda_3 \wedge \eta^3 + d\lambda_4 \wedge \eta^4 + \lambda_2\,\phi \wedge \eta^1 + (\lambda_2 I - 1)\,\phi \wedge \eta^2 + \lambda_3\,\eta^1 \wedge \eta^2 \\
& + \left(-(T^0_{02} + T^2_{03})\cos\Theta\sin\Theta + \lambda_2 T^2_{03} + \lambda_3 I\right)\phi \wedge \eta^3 \\
& + \left(T^1_{04} + \lambda_2 T^2_{04} + \lambda_3 T^3_{04} + (1 + \cos^2\Theta)\lambda_4 I\right)\phi \wedge \eta^4 \\
& + \left(B_1\cos\Theta + \lambda_2 B_2\cos\Theta + \lambda_3 B_3\cos\Theta + \lambda_4\sin\Theta\right)\eta^1 \wedge \eta^3 \\
& + \left(T^1_{14} + \lambda_2 T^2_{14} + \lambda_3 T^3_{14} + \lambda_4(B_3\cos\Theta + B_4\sin\Theta)\right)\eta^1 \wedge \eta^4 \\
& + \left(-B_1\sin\Theta - \lambda_2 B_2\sin\Theta - \lambda_3 B_3\sin\Theta + \lambda_4\cos\Theta\right)\eta^2 \wedge \eta^3 \\
& + \left(T^1_{24} + \lambda_2 T^2_{24} + \lambda_3 T^3_{24} + \lambda_4(-B_3\sin\Theta + B_4\cos\Theta)\right)\eta^2 \wedge \eta^4 \\
& + \left(T^1_{34} + \lambda_2 T^2_{34} + \lambda_3 T^3_{34} + \lambda_4 T^4_{34}\right)\eta^3 \wedge \eta^4.
\end{aligned}
$$

(4.6)

By contracting $d\zeta$ with the vector fields dual to the coframing $\{\phi, \eta^1, \eta^2, \eta^3, \eta^4, d\lambda_2, d\lambda_3, d\lambda_4\}$ on $\mathcal{Z}$, we find that subject to the condition $\tilde{\gamma}^*\eta^1 \neq 0$, the requirement that $\tilde{\gamma}' \lrcorner d\zeta = 0$ is equivalent to the condition that $\tilde{\gamma}$ is an integral curve of the system

$$
\begin{aligned}
\mathcal{J} = \{ & \eta^2, \eta^3, \eta^4, \phi - \lambda_3\eta^1, \\
& d\lambda_3 - \left(-B_1\cos\Theta + ((T^0_{02} + T^2_{03})\cos\Theta\sin\Theta - B_3\cos\Theta)\lambda_3 - (\sin\Theta)\lambda_4 - I\lambda_3^2\right)\eta^1, \\
& d\lambda_4 - \left(-T^1_{14} - (T^1_{04} + T^3_{14})\lambda_3 - (B_3\cos\Theta + B_4\sin\Theta)\lambda_4 - T^3_{04}\lambda_3^2 - (1 + \cos^2\Theta)I\lambda_3\lambda_4\right)\eta^1 \}
\end{aligned}
$$

on the submanifold $\mathcal{Y} \subset \mathcal{Z}$ defined by $\lambda_2 = 0$. Curves satisfying this requirement project to critical curves of the functional $\hat{\mathcal{L}}$ among integral curves of $\hat{\mathcal{I}}$ on $\mathcal{B}_4$, and thus to local minimizers of the sub-Finsler length functional $\mathcal{L}$ on $\mathcal{X}$. According to [5], every regular local minimizer arises in this way. Setting $\lambda = \lambda_3$, $\mu = \lambda_4$ yields the theorem. $\qquad \square$

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We will call a regular, unit-speed horizontal curve $\gamma : [a, b] \to \mathcal{X}$ a *regular sub-Finsler geodesic* if it has a lift to an integral curve of $\mathcal{J}$ on $\mathcal{Y}$. When γ has unit speed, it lifts to an integral curve of $\mathcal{J}$ if and only if it satisfies the *geodesic equations*

$$(4.7) \qquad \eta^1 = ds, \ \eta^2 = 0, \ \eta^3 = 0, \ \eta^4 = 0, \ \phi = \lambda \, ds,$$

$$d\lambda = \left(-B_1 \cos\Theta + ((T_{02}^0 + T_{03}^2)\cos\Theta\sin\Theta - B_3\cos\Theta)\lambda - (\sin\Theta)\mu - I\lambda^2\right) ds,$$

$$d\mu = \left(-T_{14}^1 - (T_{04}^1 + T_{14}^3)\lambda - (B_3\cos\Theta + B_4\sin\Theta)\mu - T_{04}^3\lambda^2 - (1 + \cos^2\Theta)I\lambda\mu\right) ds.$$

5. Rigid curves

Now we turn to the rigid curves. In [7], Sussmann proved the following theorem:

Theorem 5.1 (Sussmann). *Every locally rigid curve in a sub-Riemannian Engel manifold is locally uniquely optimal.*

In other words, every sufficiently short segment of a rigid curve in a sub-Riemannian Engel manifold is the unique length-minimizing horizontal curve among all horizontal curves with the same endpoints. We have the analogous theorem for *symmetric* sub-Finsler Engel manifolds:

Theorem 5.2. *If $(\mathcal{X}, D, F)$ is a symmetric sub-Finsler manifold (i.e., if $F(-v) = F(v)$ for all $v \in D$), then every locally rigid curve in $\mathcal{X}$ is locally uniquely optimal.*

We conjecture that Theorem 5.2 remains true without the symmetry assumption, but this assumption plays a key role in the proof.

Proof. For each $x \in \mathcal{X}$, consider the indicatrix $\Sigma_x \subset D_x$. By the symmetry assumption, Σ_x is a closed, strongly convex curve which is symmetric about the origin in D_x. Let $(\Sigma_1)_x \subset D_x$ be an ellipse centered at the origin, with the properties that:

- $(\Sigma_1)_x$ is tangent to Σ_x at the points corresponding to the rigid directions, i.e., at $\Theta = 0$ and $\Theta = \pi$.
- Σ_x minus the two points of tangency is contained within the interior of $(\Sigma_1)_x$.

There is a 1-parameter family of such ellipses, as shown in Figure 1. These ellipses can be
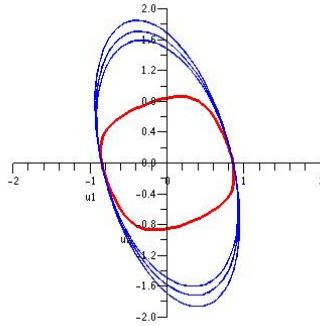

FIGURE 1. Ellipses enclosing Σ_x

chosen at each point $x \in \mathcal{X}$ so that the 3-manifold $\Sigma_1 = \underset{x \in \mathcal{X}}{\cup}(\Sigma_1)_x$ is a smooth sub-Riemannian

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structure on the Engel manifold $(\mathfrak{X}, D)$. Moreover, the corresponding sub-Riemannian metric g has the property that for any horizontal vector $\mathbf{v} \in D$,

$$|\mathbf{v}|_g \leq |\mathbf{v}|_F,$$

with equality if and only if $\mathbf{v}$ is tangent to a rigid direction.

Now, let $\gamma : [a, b] \to \mathfrak{X}$ be a rigid curve, and let $\tilde{\gamma} : [a, b] \to \mathfrak{X}$ be any other horizontal curve satisfying $\tilde{\gamma}(a) = \gamma(a)$, $\tilde{\gamma}(b) = \gamma(b)$. Let $\mathcal{L}_F(\gamma)$ denote the length of γ in the sub-Finsler metric F, and $\mathcal{L}_g(\gamma)$ the length of γ in the sub-Riemannian metric g. We have:

$$\begin{aligned}
\mathcal{L}_F(\gamma) &= \int_a^b |\gamma'(t)|_F \, dt \\
&= \int_a^b |\gamma'(t)|_g \, dt \quad \text{(because } \gamma'(t) \text{ is a rigid direction)} \\
&= \mathcal{L}_g(\gamma) \\
&\leq \mathcal{L}_g(\tilde{\gamma}) \quad \text{(by Theorem 5.1)} \\
&= \int_a^b |\tilde{\gamma}'(t)|_g \, dt \\
&\leq \int_a^b |\tilde{\gamma}'(t)|_F \, dt \\
&= \mathcal{L}_F(\tilde{\gamma}).
\end{aligned}$$

As in Theorem 5.1, we have equality only if and only if $\tilde{\gamma}$ is a reparametrization of γ. This completes the proof. $\square$

We close this section by observing that, although the derivation of the geodesic equations (4.7) is only valid for regular curves, it may happen that the rigid curves formally satisfy these equations as well. The lift $\hat{\gamma} : [a, b] \to \Sigma$ of a rigid curve γ in $\mathfrak{X}$ has the property that $\hat{\gamma}([a, b])$ lies in either the locus $\{\Theta = 0\}$ or $\{\Theta = \pi\}$. In particular, Θ is constant along $\hat{\gamma}$, so $\hat{\gamma}$ is an integral curve of the system $\{\eta^2, \eta^3, \eta^4, \phi\}$ on Σ. If $\hat{\gamma}$ is a solution of (4.7), then $\lambda = \sin \Theta = 0$ along $\hat{\gamma}$. But then

$$d\lambda = \mp B_1 ds = 0$$

as well (the choice of sign depends on whether $\Theta = 0$ or $\Theta = \pi$); thus $B_1 = 0$ along $\hat{\gamma}$. In this case, the equation for $d\mu$ along $\hat{\gamma}$ simplifies to

$$d\mu = \left(-T_{14}^1 \mp B_3 \mu\right) ds.$$

This equation has a 1-parameter family of solutions $\mu(s)$ along $\hat{\gamma}$. Unlike in the case of a regular curve, varying the choice of μ has no impact on the curve.

This discussion yields:

Theorem 5.3. *A rigid curve $\gamma : [a, b] \to \mathfrak{X}$ formally satisfies the geodesic equations (4.7) if and only if $B_1 \equiv 0$ along the lifted curve $\hat{\gamma} : [a, b] \to \Sigma$.*

6. An example

Example 6.1. Let us revisit the kinematic penny of Example 1.8. We will modify the sub-Riemannian structure described there according to the notion that *curvature is costly*: in other words, it takes more effort to steer the wheel in a tight circle with little forward or

backward motion than to steer it in a wide arc. Since the curvature of the projection $\bar{\gamma}$ is given by $\kappa = \frac{\dot{\psi}}{\dot{\phi}}$, this leads us to consider sub-Finsler metrics of the form

$$F = f\left(\frac{\dot{\psi}}{\dot{\phi}}\right)\sqrt{d\phi^2 + d\psi^2},$$

where f gets larger (but stays bounded) as $\left|\frac{\dot{\psi}}{\dot{\phi}}\right|$ increases. One must choose f carefully in order to ensure that the resulting F is, in fact, sub-Finsler; many choices of f lead to a non-convex indicatrix Σ_x. For instance, the function $f(\kappa) = e^{\frac{\kappa^2}{1+2\kappa^2}}$ determines a sub-Finsler metric; the graph of this function is shown in Figure 2. Figure 3 shows the resulting
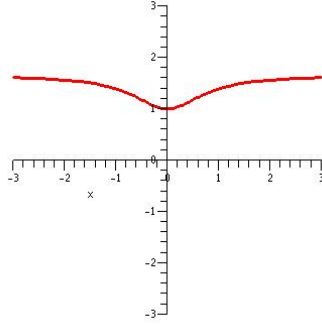
FIGURE 2. $f(\kappa) = e^{\frac{\kappa^2}{1+2\kappa^2}}$

sub-Finsler indicatrix in the $(\dot{\varphi}, \dot{\psi})$ plane, compared to the sub-Riemannian indicatrix (unit circle).
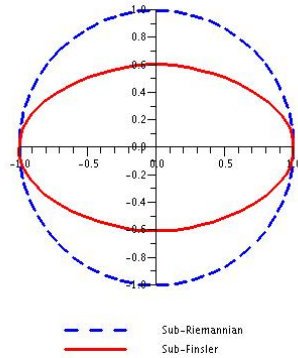
FIGURE 3. Sub-Riemannian and sub-Finsler indicatrices

In order to compare the sub-Riemannian and sub-Finsler metrics for this example, we will construct the canonical coframings and compute the geodesic equations for both structures. We will then solve these equations numerically for several different initial conditions and compare the results.

The canonical coframing on the sub-Riemannian indicatrix (unit circle) bundle $\Sigma_1 = \mathcal{X} \times S^1$, with coordinate θ on the S^1 factor, is given by

$$\alpha = -d\theta$$
$$\omega^1 = (\cos\theta)d\varphi + (\sin\theta)d\psi$$
(6.1)
$$\omega^2 = -(\sin\theta)d\varphi + (\cos\theta)d\psi$$
$$\omega^3 = -(\sin\psi)dx + (\cos\psi)dy$$
$$\omega^4 = (\cos\psi)dx + (\sin\psi)dy - d\varphi.$$

This coframing has structure equations

$$d\alpha = 0$$
$$d\omega^1 = -\alpha \wedge \omega^2$$
(6.2)
$$d\omega^2 = \alpha \wedge \omega^1$$
$$d\omega^3 = \omega^1 \wedge \omega^2 - [(\sin\theta)\omega^1 + (\cos\theta)\omega^2] \wedge \omega^4$$
$$d\omega^4 = [(\sin\theta)\omega^1 + (\cos\theta)\omega^2] \wedge \omega^3.$$

The geodesic equations (4.7) for this sub-Remannian structure are equivalent to the following system of ODEs:

$$x'(s) = \cos(\psi(s))\cos(\theta(s))$$
$$y'(s) = \sin(\psi(s))\cos(\theta(s))$$
$$\varphi'(s) = \cos(\theta(s))$$
(6.3)
$$\psi'(s) = \sin(\theta(s))$$
$$\theta'(s) = -\lambda(s)$$
$$\lambda'(s) = -\mu(s)\sin(\theta(s))$$
$$\mu'(s) = \lambda(s)\sin(\theta(s)).$$

The sub-Finsler structure Σ is obtained from Σ_1 via the construction (3.1), with

$$r(\theta) = f(\tan\theta) = e^{\frac{\tan^2\theta}{1+2\tan^2\theta}}.$$

The canonical coframing on Σ may be written in terms of the sub-Riemannian coframing (6.1) (omitting the obvious pullback notation) as follows:

$$\phi = \frac{\sqrt{r(r+r'')}}{r}\alpha$$
$$\eta^1 = r\omega^1 + r'\omega^2$$
(6.4)
$$\eta^2 = \sqrt{r(r+r'')}\omega^2$$
$$\eta^3 = \sqrt{r^3(r+r'')}\omega^3$$
$$\eta^4 = \frac{r^3(r+r'')}{\sqrt{(r\cos\theta - r'\sin\theta)^2 + r(r+r'')\sin^2\theta}}\omega^4,$$

with structure equations

$$d\phi = 0$$
$$d\eta^1 = -\phi \wedge \eta^2$$
$$d\eta^2 = \phi \wedge \eta^1 + I\phi \wedge \eta^2$$
$$d\eta^3 = \eta^1 \wedge \eta^2 + I\phi \wedge \eta^3 + (T_{14}^3\eta^1 + T_{24}^3\eta^2) \wedge \eta^4$$
$$d\eta^4 = \left[(\sin\Theta)\eta^1 + (\cos\Theta)\eta^2\right] \wedge \eta^3 + (1 + \cos^2\Theta)I\phi \wedge \eta^4,$$

where

$$I = -\frac{4\sin^3\theta\cos\theta\,(15\cos^4\theta - 32\cos^2\theta + 12)}{(8 + \cos^8\theta - 12\cos^6\theta + 26\cos^4\theta - 20\cos^2\theta)^{3/2}}$$

$$\sin\Theta = \frac{\sin\theta\sqrt{8 + \cos^8\theta - 12\cos^6\theta + 26\cos^4\theta - 20\cos^2\theta}}{\sqrt{8 + 9\cos^8\theta - 30\cos^6\theta + 38\cos^4\theta - 24\cos^2\theta}}$$

$$\cos\Theta = \frac{\cos\theta(\cos^4\theta - 2\cos^2\theta + 2)}{\sqrt{8 + 9\cos^8\theta - 30\cos^6\theta + 38\cos^4\theta - 24\cos^2\theta}}$$

$$T_{14}^3 = -\frac{\sin\theta\sqrt{8 + 9\cos^8\theta - 30\cos^6\theta + 38\cos^4\theta - 24\cos^2\theta}}{(r(\theta))^2\sqrt{8 + \cos^8\theta - 12\cos^6\theta + 26\cos^4\theta - 20\cos^2\theta}}$$

$$T_{24}^3 = -\frac{\cos\theta(\cos^4\theta - 2\cos^2\theta + 2)\sqrt{8 + 9\cos^8\theta - 30\cos^6\theta + 38\cos^4\theta - 24\cos^2\theta}}{(r(\theta))^2(8 + \cos^8\theta - 12\cos^6\theta + 26\cos^4\theta - 20\cos^2\theta)}.$$

The geodesic equations (4.7) for this sub-Finsler structure are equivalent to the following system of ODEs:

$$x'(s) = \frac{\cos(\psi(s))\cos(\theta(s))}{r(\theta(s))}$$

$$y'(s) = \frac{\sin(\psi(s))\cos(\theta(s))}{r(\theta(s))}$$

$$\varphi'(s) = \frac{\cos(\theta(s))}{r(\theta(s))}$$

(6.5)
$$\psi'(s) = \frac{\sin(\theta(s))}{r(\theta(s))}$$

$$\theta'(s) = -\lambda(s)\frac{\sqrt{r(\theta(s))}}{\sqrt{r(\theta(s)) + r''(\theta(s))}}$$

$$\lambda'(s) = -\mu(s)\sin\Theta(\theta(s)) - \lambda(s)^2 I(\theta(s))$$

$$\mu'(s) = -\lambda(s)T_{14}^3(\theta(s)) - \lambda(s)\mu(s)(1 + \cos^2\Theta(\theta(s)))I(\theta(s)).$$

In order to compare these two metrics, we will integrate the geodesic equations (6.3) and (6.5) numerically for several different choices of initial values for the parameters. In all

the figures that follow, we plot the images of the appropriate paths $\bar{\gamma}$ in the xy plane. In each case, we will choose initial values $(x(0), y(0), \varphi(0), \psi(0)) = (0, 0, 0, \frac{\pi}{6})$; we will see how varying the initial values of θ, λ, and μ changes the trajectories.

- *Rigid curves.* For both of these metrics, $B_1 \equiv 0$, and so the rigid curves formally satisfy the geodesic equations. Rigid curves may be obtained by choosing initial values

$$\big(\theta(0), \lambda(0), \mu(0)\big) = \big(0, 0, 0\big).$$

As expected, both paths in the xy-plane are straight lines (Figure 4). Since there is no curvature, these paths are traced at the same speed in both metrics.
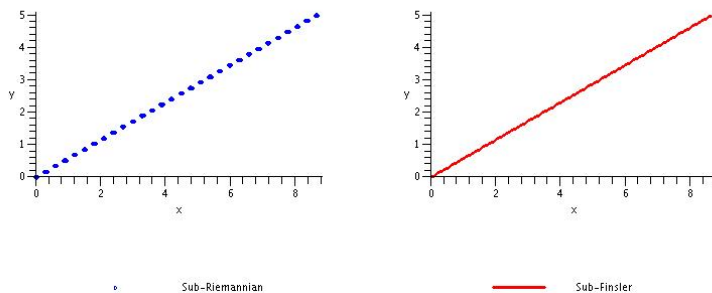
FIGURE 4. $\big(\theta(0), \lambda(0), \mu(0)\big) = \big(0, 0, 0\big)$

- *Nonzero $\theta(0)$.* Choosing a nonzero initial value for θ leads to a nonzero value for $\psi'(s)$, and hence introduces some curvature. If we choose

$$\big(\theta(0), \lambda(0), \mu(0)\big) = \big(\tfrac{\pi}{4}, 0, 0\big),$$

then both paths trace the same circle in the xy plane. As a result of the nonzero curvature, the sub-Finsler geodesic traces the curve more slowly than the sub-Riemannian one (Figure 5).
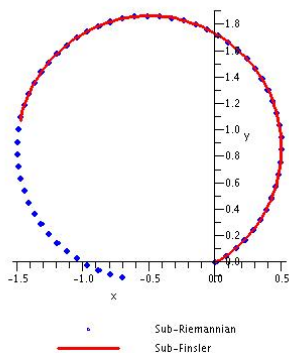

FIGURE 5. $\big(\theta(0), \lambda(0), \mu(0)\big) = \big(\tfrac{\pi}{4}, 0, 0\big)$

- *Nonzero* $\lambda(0)$. Choosing a nonzero initial value for λ leads to a nonzero value for $\theta'(s)$, and hence for $\psi''(s)$. Choosing

$$\big(\theta(0), \lambda(0), \mu(0)\big) = \big(0, 0.2, 0\big)$$

yields the paths shown in Figure 6. Here we see the effect of our hypothesis that curvature is costly: the sub-Finsler geodesic does not curve as sharply or travel as far as the sub-Finsler one.
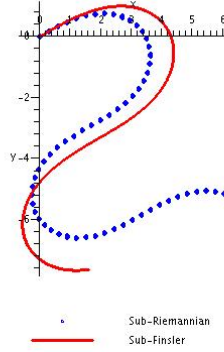

FIGURE 6. $\big(\theta(0), \lambda(0), \mu(0)\big) = \big(0, 0.2, 0\big)$

- *Nonzero* $\mu(0)$. Choosing a nonzero initial value for μ leads to a nonzero value for $\lambda'(s)$ – unless we choose $\theta(0) = \lambda(0) = 0$, in which case we obtain the rigid curves again). In order to avoid this case, we choose

$$\big(\theta(0), \lambda(0), \mu(0)\big) = \big(\tfrac{\pi}{4}, 0, 0.2\big).$$

This yields the paths shown in Figure 7. Observe that the sub-Finsler geodesic reverses course rather than curve as tightly as the sub-Riemannian one. (We also note that while the sub-Finsler path $\bar{\gamma}$ contains cusps, the path γ in the state space $\mathcal{X}$ is in fact smooth.)
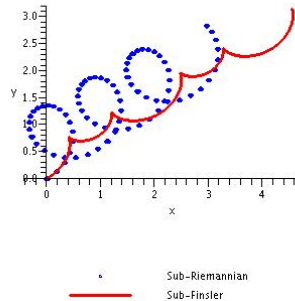

FIGURE 7. $\big(\theta(0), \lambda(0), \mu(0)\big) = \big(\tfrac{\pi}{4}, 0, 0.2\big)$

If we choose the slightly larger value of $\mu(0) = 0.5$, then both the sub-Riemannian and sub-Finsler geodesics reverse course, but the sub-Riemannian one still curves much more sharply than the sub-Finsler one (Figure 8).
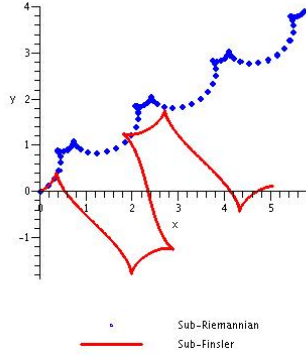
FIGURE 8. $\left(\theta(0), \lambda(0), \mu(0)\right) = \left(\frac{\pi}{4}, 0, 0.5\right)$

- *Nonzero $\lambda(0)$ and $\mu(0)$.* For yet another example, choosing
$$\left(\theta(0), \lambda(0), \mu(0)\right) = \left(0, 0.2, 0.2\right)$$
yields the paths shown in Figure 9. This choice shows how geodesics for the two metrics can exhibit very different behavior, and illustrates once again how the sub-Finsler metric is more averse to following a tightly curved path than the sub-Riemannian one.
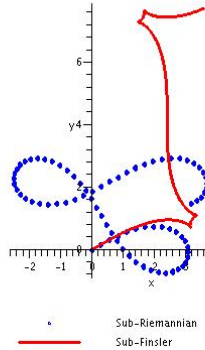

FIGURE 9. $\left(\theta(0), \lambda(0), \mu(0)\right) = \left(0, 0.2, 0.2\right)$

7. CONCLUSION

Example 6.1 shows how a relatively simple modification of a sub-Riemannian metric can change its behavior significantly. There are other natural sub-Finsler metrics to consider; for instance, a wheel which requires more energy to move backwards than forwards would lead to a non-symmetric sub-Finsler structure. Allowing this more general structure opens the door to consideration of a much wider class of control problems than those described by sub-Riemannian geometry.

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