Calculus I (Math 241) – Test 1

Problem 1. [5 Points] Suppose that

$$\left(x^2 + y^2\right)^2 = 5x^2y.$$

Find dy/dx in terms of x and y.

Solution: Differentiating both sides of the equation we find:

$$2\left(x^2 + y^2\right)[2x + 2yy'] = 10xy + 5x^2y'.$$

Bringing all terms with a factor y' to the left side of the equation, and all terms without such a factor to the right, we obtain:

$$\left[4y(x^2 + y^2) - 5x^2\right] y' = 10xy - 4x(x^2 + y^2).$$

and

$$y' = \frac{10xy - 4x(x^2 + y^2)}{4y(x^2 + y^2) - 5x^2}.$$

Problem 2. [5 Points] Find

$$\lim_{x \to 0} \frac{\tan x}{3x}.$$

Solution: We calculate:

$$\lim_{x \to 0} \frac{\tan x}{3x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{3\cos x} = 1 \cdot \frac{1}{3} = \frac{1}{3}.$$

Problem 3. [10 Points] Show that the equation

$$x = 2\sin x$$

has a solution with $x \in [\pi/2, \pi]$. Name the theorem that you apply, and tell why its assumptions hold. Hint: Equivalently you may show that the function $f(x) = x - 2\sin x$ has a zero in the interval $[\pi/2, \pi]$.

Solution: As the difference of two continuous functions, $f(x) = x - 2\sin x$ is continuous, $f(\pi/2) = \frac{\pi}{2} - 2 < 0$, and $f(\pi) = \pi > 0$. The number 0 lies between these two values. The Intermediate Value Theorem tells us that there is an x between $\frac{\pi}{2}$ and π, so that $f(x) = 0$, or equivalently, $x = 2\sin x$.

Problem 4. [10 Points] Use first principles (the definition of the derivative and the computation of the limit of the appropriate difference quotient) to find the derivative of the function $f(x) = \frac{1}{\sqrt{x}}$.

Solution: According to the definition, for $a > 0$:

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x-a} \\
&= \lim_{x\to a} \frac{\frac{\sqrt{a}-\sqrt{x}}{\sqrt{x}\sqrt{a}}}{x-a} \\
&= \lim_{x\to a} \frac{(\sqrt{a}-\sqrt{x})(\sqrt{a}+\sqrt{x})}{\sqrt{x}\sqrt{a}(x-a)(\sqrt{a}+\sqrt{x})} \\
&= \lim_{x\to a} \frac{a-x}{\sqrt{x}\sqrt{a}(x-a)(\sqrt{a}+\sqrt{x})} \\
&= \lim_{x\to a} \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})} \\
&= \frac{-1}{2a^{3/2}} = -\frac{1}{2}a^{-\frac{3}{2}}.
\end{aligned}$$

We calculated that

$$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}.$$

Problem 5. [10 Points] Find the equation of the tangent line to the graph of the function $f(x) = x\tan x$ at the point where $x = \pi/6$.

Solution: We calculate:

$$f(\pi/6) = \frac{\pi}{6}\tan\frac{\pi}{6} = \frac{\pi}{6}\frac{1/2}{\sqrt{3}/2} = \frac{\pi}{6\sqrt{3}}$$

and

$$f'(x) = \tan x + x\sec^2 x \quad \& \quad f'(\pi/6) = \frac{1}{\sqrt{3}} + \frac{\pi}{6}\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{1}{\sqrt{3}} + \frac{2\pi}{9}$$

The equation for the tangent line is

$$t(x) = \left(\frac{1}{\sqrt{3}} + \frac{2\pi}{9}\right)(x - \frac{\pi}{6}) + \frac{\pi}{6\sqrt{3}}$$

Problem 6. [10 Points] A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m (meter) higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s (meter per second), how fast is the boat approaching the dock when it is 8 m from the dock?

Solution: You have a right triangle. The hypotenuse is the rope. One side drops vertically from the pulley to the water, and the other side is the horizontal path along which the bow of the boat travels. Let $r(t)$ denote the length of the rope, bow to pulley, and x the distance between the bow and the dock. Then

$$r = \sqrt{x^2+1} \quad \& \quad \frac{dr}{dx} = \frac{x}{\sqrt{x^2+1}} \quad \& \quad \left.\frac{dr}{dx}\right|_{x=8} = \frac{8}{\sqrt{65}}.$$

The equation for the related rates is:

$$\frac{dr}{dx}\frac{dx}{dt} = \frac{dr}{dt}.$$

As stated in the problem: $dr/dt = 1$. We calculated for the specified moment that $dr/dx = 8/\sqrt{65}$. It follows that at the given moment the boat approaches the dock with a speed of

$$\frac{dx}{dt} = \frac{\sqrt{65}}{8} \sim 1.0078 \ (m/s).$$

Problem 7. [10 Points] Use approximation by differentials to find an approximate value of the cosine function at $61^\circ = \frac{61\pi}{180}$ rad.

Solution: Using the general formula that $f(x) \sim f(a) + f'(a)(x-a)$ with $f(x) = \cos x$, $f'(x) = -\sin x$, and $a = \pi/3$, we find:

$$\cos(61^\circ) = \cos\left(\frac{61\pi}{180}\right) \sim \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{180} = \frac{1}{2} - \frac{\pi\sqrt{3}}{360}$$

Problem 8. [10 Points] Use Newton's method to find a zero of the function $f(x) = x^3 - 3$, approximately. As initial guess, use $x_0 = 1$. Improve this guess twice.

Solution: The general formula for the (hopefully) improved guess is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3}{3x_n^2} = \frac{2x_n^3 + 3}{3x_n^2}$$

We see that $x_1 = 5/3$ and $x_2 = 331/225$.