Calculus I (Math 241) – Final

Problem 1. [15 Points] Calculate the derivatives:

$$\frac{d}{dx}\left(\frac{x+1}{x^2+1}\right) \quad \text{and} \quad \frac{d}{dx}(2x+\sin^2 x)^3 \quad \text{and} \quad \frac{d}{dx}\sec(\sqrt{x^2+x})$$

Problem 2. [10 Points] Differentiate $f(x) = x^2$ using first principles. I.e., express $f'(a)$ as a limit of a difference quotient and work out the limit. (Make sure to show every step. This may take you 6 lines!)

Problem 3. [15 Points] The point $(x, y) = (2, 3)$ lies on the hyperbola

$$\frac{x^2}{\frac{8}{3}} - \frac{y^2}{18} = 1.$$

1. Find the equation of the tangent line to the hyperbola through this point.
2. Suppose $(2.2, Y)$ is a point on the hyperbola. Use approximation of differentials to find an approximate value for Y.

Problem 4. [5 Points] Suppose $f(x)$ is a function and c is an interior point of its domain. What does it mean that $f(x)$ is continuous at c? (Give a precise definition.)

Problem 5. [20 Points] You are building a box. The base is square, and it has a bottom but no top. Find the side length and height of the box with the largest volume if you use $A = 100$ square feet of material.

Problem 6. [10 Points] Sketch the function $f(x) = |x|$ on the interval $[-1, 1]$. In the same coordinate system sketch an anti-derivative $F(x)$ of $f(x)$, so that $F(0) = 2$.

Problem 7. [20 Points] Discuss the function

$$f(x) = x(x-1)(x-3) = x^3 - 4x^2 + 3x.$$

Specifically, answer the following questions:

1. On which intervals is the function positive, resp., negative.

2. Find the critical points of the function and decide on which intervals the function is increasing, resp., decreasing.

3. Decide where the function has local maxima, resp., minima.

4. Find the inflections points and decide on which intervals the function is concave up, resp., down.

5. Sketch the graph in accordance with the information obtained above.

Problem 8. [25 Points] Find the indefinite integrals

$$\int \cos x \; dx \qquad \int x\sqrt{x} \; dx \qquad \int \tan x \sec^2 x \; dx \qquad \int \frac{x}{\sqrt{x^2+9}} \; dx \qquad \int \frac{dx}{x^2+9}.$$

Problem 9. [15 Points] A particle moves along a line. We measure time in seconds. The scale on the line uses meters as units. We denote the position on the line by $x(t)$. At time $t = 0$ the position of the particle is $x(0) = 3$. The initial velocity is $v(0) = -4$ (meter per second). The acceleration is constant, $a(t) = 2$.

1. Find the velocity of the particle at any time, i.e., find $v(t)$.

2. Find the position of the particle at any time, i.e., find $x(t)$.

3. How far from its intial position is the particle after 3 seconds, and how far did the particle travel in the first three seconds?

Problem 10. [10 Points] Consider the function $y = \sin x$ on the interval $[0, \pi/2]$. Partition the interval, breaking it up into the intervals $[0, \pi/4]$ and $[\pi/4, \pi/2]$. Find the Riemann sums using the points:

1. left endpoints.

2. right endpoints.

3. $x_1^* = \pi/6$ in the first and $x_2^* = \pi/3$ in the second interval.