

Calculus I (Math 241) – Final

Problem 1. [9 Points] Calculate the derivatives:

$$\frac{d}{dx} (x^3 \sin^2(2x + 1)) \quad \text{and} \quad \frac{d}{dx} \left[\frac{x^2 + 1}{2 + \sin x} \right] \quad \text{and} \quad \frac{d}{dx} \sec(1 + \cos^2 x)$$

Problem 2. [16 Points] State in precise mathematical term what it means that

$$\lim_{x \rightarrow a} H(x) = L.$$

This includes a statement telling what $H(x)$ is, and where it needs to be defined.

Differentiate $f(x) = x^3$ using first principles. I.e., express $f'(a)$ as a limit of a difference quotient and work out the limit. (Make sure to show every step. This may take you 6 lines!)

Problem 3. [10 Points] Use $\sqrt[4]{16} = 2$ and approximation by differentials to find an approximate value for $\sqrt[4]{18}$.

Problem 4. [10 Points] Find the equation of the tangent line $t(x)$ to the graph of the function $f(x) = \cos^2 x$ at $x = \pi/3$.

Problem 5. [15 Points] Starting out with a disk a radius R , form a round open container (no top) whose sides are straight up. You will have to pleat the edge as you bend it up to form the sides. Find the dimensions of the container so that its volume is maximal.

Problem 6. [5 Points] State Cauchy's Mean Value Theorem. This includes the assumptions and conclusion.

Problem 7. [30 Points] Discuss the function

$$f(x) = x(x - 3)^2 = x^3 - 6x^2 + 9x.$$

Specifically, answer the following questions:

1. On which intervals is the function positive, resp., negative.
2. On which intervals is the function increasing, resp., decreasing.
3. On which intervals is the function concave up, resp., concave down.

4. Find the critical points of the function and decide whether they are local maxima or minima.
5. Find the inflections points of the function.
6. Find the absolute extrema of the function on the interval $[-1, 3.2]$.
7. Sketch the graph in accordance with the information obtained above.

Problem 8. [15 Points] Find the indefinite integrals

$$\int x^2 \sqrt{x+1} dx \quad \text{and} \quad \int \sin^2 x dx \quad \text{and} \quad \int \frac{dx}{x^2 + 25}.$$

Problem 9. [15 Points] Consider the function

$$f(x) = x^5 - 7x^2 - 2x - 1.$$

Show that this function has a root in the interval $[2, 3]$. (You are expected to quote a theorem, make sure to verify its assumptions and apply its conclusion explicitly.)

A fairly good guess for this root is $x_0 = 2$. Use Newton's method to improve this guess once.

Problem 10. [20 Points] Compute the intersection points of the curves $y = x - 1$ and $y = -(x - 3)^2 + 4$ and sketch the region Ω enclosed by these two curves.

1. Consider that solid of revolution of Ω around the x -axis. Express its volume as an integral and compute it.
2. Consider that solid of revolution of Ω around the y -axis. Express its volume as an integral, but **do not compute it**.
3. Consider that solid of revolution of Ω around the axis $y = 5$. Express its volume as an integral, but **do not compute it**.