Calculus I (Math 241) – Final

Problem 1. [16 Points] Calculate the derivatives of the following functions:

$$f(x) = \sin^3(2x-5) \quad g(x) = \sqrt{1+\sec(2x)}, \quad h(x) = \frac{\sin(2x)}{\tan(3x)}, \quad t(x) = \sqrt{\frac{1+x}{1-x}}.$$

Problem 2. [16 Points] Calculate the integrals:

$$\int x(x-1)^\pi \, dx, \quad \int \sec^2 x \tan^2 x \, dx, \quad \int \frac{dt}{16+t^2}, \quad \int_0^{\frac{\pi}{3}} \sin^2 x \, dx.$$

Problem 3. [20 Points] Discuss the function

$$f(x) = x^3 - 2x^2 - 3x.$$

Specifically, address the following:

1. Find the intercepts and the intervals on which the function is positive, resp., negative.
2. Find the critical points and the intervals on which the function is increasing, resp., decreasing.
3. Find the intervals on which the function is concave up, resp., concave down.
4. Find the local maxima and minima and the inflection points.
5. Sketch the graph in accordance with the information obtained above.

Problem 4. [14 Points] Use first principles to find $f'(a)$ if $f(x) = x^4$.

Problem 5. [15 Points] Consider a curve that is defined by the equation

$$y' = y(4-y)x \qquad \text{with} \quad y(2) = 3.$$

Find an equation for the tangent line to the curve at the point $(x, y) = (2, 3)$. Find y'' in general as well as y'' at the point $(x, y) = (2, 3)$.

Problem 6. [20 Points] In the Sun Flower Memorial Biathalon you run along Blue Moon River and then swim across it. The finish line is 10 miles down stream and on the opposite side from the starting point. The river is one mile wide. You can run 6 miles and swim 1 mile per hour. You are allowed to start swimming at any point along the way. Express your finishing time $T(x)$ as a function of the point where you begin to swim. Indicate in a sketch what x stands for in your solution. Minimize T.

Problem 7. [10 Points] Apparently, $x = 2$ is not very far from $\sqrt[3]{10}$, or 2^3 is approximately 10. Use Newton's method to improve on this guess once.

Problem 8. [8 Points] State the Fundamental Theorem of Calculus. This includes the assumptions and conclusion.

Problem 9. [13 Points] Calculate the Riemann sum for the following situation. The function is $f(x) = x^2$, and the interval is $[0, 2]$. Use the points $x_0 = 0$, $x_1 = 1$, $x_2 = \frac{3}{2}$ and $x_3 = 2$ to partition the interval into 3 subintervals. Choose the midpoint in each of the subintervals as distinguished point.

Problem 10. [18 Points] Sketch the region above the line $y = \frac{1}{2}$ and below the function $f(x) = \sin x$. Denote by Ω the part over the interval $[0, \pi]$.

1. Find the area of the region Ω.

2. Find the volume of the solid of revolution if Ω is revolved around the x-axis.

3. Set up an integral that computes the volume of the solid of revolution if Ω is revolved around the y-axis.