Calculus II (Math 242) – Test 2

Problem 1. [10 Points] Find

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n-1)}.$$

Hint: Use a partial fraction decomposition.

Problem 2. [10 Points] Decide for which values of p the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges, and for which it diverges. Name the test which you apply, and show how you apply it.

Problem 3. [10 Points] Evaluate the series and find its radius of convergence:

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + 2x^7 + \cdots.$$

(Alternatingly, the coefficients are 1 and 2.)

Problem 4. [8 Points] Find the power series expansion of $\sin(x^2)$.

Problem 5. [12 Points] Find the power series expansion of $f(x) = \frac{1}{(1-x)^3}$. Express the result as a sum, and also list the first three terms explicitly.

Problem 6. [20 Points] Recall that $\cosh x = [e^x + e^{-x}]/2$, $\cosh' x = \sinh x$, $\sinh' x = \cosh x$, $\cosh 0 = 1$ and $\sinh 0 = 0$.

1. Find the Taylor series of $\cosh x$.
2. Use the first four non-zero terms of the series to get an approximate value for $\cosh(1/2)$. Call this approximate value B.
3. Use the remainder formula to find an estimate for $|B - \cosh(1/2)| = E$. For your numerical estimate you may use that $\cosh(1/2) < 1.3$ and $\sinh(1/2) < .53$.

Problem 7. [15 Points] Find the Taylor series of $f(x) = \sqrt{9 + x^2}$. Provide the first five nonzero terms explicitly, as well as the general expression. What is the radius of convergence?

Problem 8. [15 Points] Write out the power series expansion of $\sin x$ around $a = \pi/6$. How may terms do you need to calculate $\sin(31^\circ)$ accurately up to 4 decimal places?

Solutions (Exam 2, Math 242)

#1

$$\frac{1}{(2n+1)(2n-1)} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] = \frac{1}{2}\left[\frac{(2n+1)-(2n-1)}{(2n+1)(2n-1)}\right]$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n-1)} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{-1} - \frac{1}{1}\right) + \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots\right]$$

$$= \frac{-1}{2}.$$

#2 We use the integral comparison test.

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{A\to\infty} \ln(\ln x)\Big|_2^A = \infty \text{ (divergent}$$

$p \neq 1$ $$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \lim_{A\to\infty} \frac{1}{1-p}\frac{1}{(\ln x)^{p-1}}\Big|_2^A = \begin{cases} \text{finite} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$

The sum $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, as well as above integral, converge if $p > 1$ and diverge if $p \leq 1$.

(Observe that $\frac{1}{x(\ln x)^p}$ decreases eventually if $p \geq 0$; if $p < 0$, then $\frac{1}{x} < \frac{1}{x(\ln x)^p}$ (eventually), and the series converges by a comparison with $\sum \frac{1}{n}$.)

#3 $$1+2x+x^2+2x^3+x^4+2x^5+\dots$$

$$= (1+2x)\,(1+x^2+x^4+\dots)$$

$$= \frac{1+2x}{1-x^2}$$

The result holds within the radius of convergence $R=1$.

#4 $$\sin u = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)!} \quad ; \sin(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

#5 $$f(x) = \frac{1}{(1-x)^3} = \frac{d^2}{dx^2}\,\frac{1}{2}\cdot\frac{1}{1-x}$$

$$= \frac{1}{2}\,\frac{d^2}{dx}\left(1+x+x^2+x^3+\dots\right)$$

$$= \frac{1}{2}\left[2\cdot 1+3\cdot 2\,x+4\cdot 3x^2+\dots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)\,x^n = \sum_{n=0}^{\infty}\binom{-3}{n}x^n$$

(You may also use the "Binomial Series".)

#6. (1) $$\cosh x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = 1+\frac{x^2}{2}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots$$

(2) $$\cosh\left(\tfrac{1}{2}\right) \sim 1+\frac{1}{8}+\frac{1}{16\cdot 24}+\frac{1}{64\cdot 720}$$

$$= \frac{46080+5760+120+1}{46080}$$

$$= \frac{51961}{42080}$$

(3) $E < \frac{.6}{7!}\left(\frac{1}{2}\right)^7 = \frac{.6}{5040 \cdot 128} \sim 10^{-6}$

Here $.6 > \max\limits_{0 \le x \le 1/2} |\cosh^{(7)}(x)|$

We could use $R_7 \le \frac{1.2}{8!}\left(\frac{1}{2}\right)^8 \sim \frac{1.2}{40{,}000 \cdot 256} \sim 10^{-7}$

#17 $f(x) = \sqrt{9+x^2}$; $u = x^2/9$

$h(u) = \sqrt{9+9u} = 3\sqrt{1+u}$

$= 3\left(1 + \frac{1}{2}u - \frac{1}{2^2 \cdot 2!}u^2 + \frac{3}{2^3 \cdot 3!}u^3 - \frac{3 \cdot 5}{2^4 \cdot 4!}u^4 + \right)$

$= 3\left(1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 - \frac{5}{128}u^4 + \right)$

$f(x) = 3\left(1 + \frac{x^2}{18} - \frac{x^4}{8 \cdot 81} + \frac{x^6}{16 \cdot 729} - \frac{5x^8}{128 \cdot 6561} + \right)$

$= 3 \sum_{n=0}^{\infty} \frac{\binom{1/2}{n}}{9^n} x^{2n}$

$h(u) = 3\sqrt{1+u}$

$h'(u) = 3 \cdot \frac{1}{2}(1+u)^{-1/2}$

$h''(u) = 3 \cdot \frac{1}{2}\left(-\frac{1}{2}\right)(1+u)^{-3/2}$

$h'''(u) = 3 \cdot \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(1+u)^{-5/2}$

The Taylor series of h converges for $|u| < 1$, so for the one for f we need $|x^2/9| < 1$ or $|x| < 3$.

#8 $\sin(\pi/6) = 1/2$

$\sin'(\pi/6) = \sqrt{3}/2$

$\sin''(\pi/6) = -1/2$

$\sin'''(\pi/6) = -\sqrt{3}/2$ (after this it repeats)

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2\cdot 2}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2\cdot 3!}\left(x - \frac{\pi}{6}\right)^3$$

$+ \ldots$

Set $x = \frac{31\pi}{180}$ $(= 31^\circ)$ so $x - \frac{\pi}{6} = \frac{\pi}{180}$

We want $R_n < 10^{-4}$.

$$R_2 \leq \frac{1}{3!}\left(\frac{\pi}{180}\right)^3 \sim \frac{1}{6\cdot 60\cdot 60\cdot 60} \sim \frac{1}{1,296,000}$$

(good enough!)

$$R_1 \leq \frac{1}{2!}\left(\frac{\pi}{180}\right)^2 \sim \frac{1}{2\cdot 60\cdot 60} \approx \frac{1}{7200}$$

(not good enough, a more careful estimate might tell, even this is ok.)