INSTRUCTIONS: Write legibly. Indicate your answer clearly. Show all work; explain your answers. Answers with work not shown might be worth zero points. No calculators, cell phones, or cheating.

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(15) 1. A slender rod of constant density $\delta$ lies along the line segment $C$ in the $yz$–plane, parametrized by $r(t) = (0, t, 2t - 2t^2)$, $0 \leq t \leq 1$. Find $I_x$, the moment of inertia about the $x$–axis.

Compare with Problem 28 from Section 15.1. Note that the velocity is $|r'(t)| = |(0, 1, -2)| = \sqrt{5}$. The moment of inertia is:

$$I_x = \delta \int_C (y^2 + z^2) \, ds$$

$$= \delta \int_0^1 (t^2 + (2 - 2t)^2) \sqrt{5} \, dt$$

$$= \delta \sqrt{5} \int_0^1 (5t^2 - 8t + 4) \, dt$$

$$= \frac{5\sqrt{5}}{3} \delta$$

(15) 2. Find the work $W$ done by the vector field $F(x, y, z) = (2y, 3x, x + y)$ over the curve $C$ parametrized by $r(t) = (\cos t, \sin t, t/6)$, for $0 \leq t \leq 2\pi$, in the direction of increasing $t$.

Compare with Problems 14 from Section 15.2. We calculate

$$W = \int_0^{2\pi} F \cdot \frac{dr}{dt} \, dt$$

$$= \int_0^{2\pi} (2\sin t, 3\cos t, \cos t + \sin t) \cdot (-\sin t, \cos t, 1/6) \, dt$$

$$= \int_0^{2\pi} -2\sin^2 t + 3\cos^2 t + \frac{\cos t + \sin t}{6} \, dt$$

$$= \frac{\pi}{3}$$

(15) 3. Find the flux of $F(x, y) = (x + y, x^2 + y^2)$ outward across the triangle with vertices $(1, 0)$, $(0, 1)$, and $(-1, 0)$.

Compare with Problems 30 from Section 15.2. The perimeter of the triangle decomposed into three smooth pieces, $C_R$ at the right, $C_L$ at the left, and $C_B$ at the bottom. We calculate the flux $F_r$, $F_L$ and $F_B$ across each one of them.

Let us use first principles only. We parametrize $C_R$ by $r(t) = (1 - t, t)$ with $0 \leq t \leq 1$, going counter
clockwise around the triangle. The unit outward normal vector is \( n = \frac{1}{\sqrt{2}}(1, 1) \), \( ds = |r'(t)|dt \), and

\[
F_R = \int_0^1 F \cdot n \, ds = \int_0^1 (1, (1 - t)^2 + t^2) \cdot \frac{1}{\sqrt{2}}(1, 1) \sqrt{2} dt = \int_0^1 (1 + (1 - t)^2 + t^2) dt = 2 \int_0^1 1 - t + t^2 dt = \frac{5}{3}.
\]

For the remaining pieces of the curve, we use the formula on page 856 of the text book. We parametrize \( C_L \) by \( r(t) = (x(t), y(t)) = (-t, 1-t) \) with \( 0 \leq t \leq 1 \). Set \( M = x + y \) and \( N = x^2 + y^2 \). Then

\[
F_L = \oint_{C_L} M \, dy - N \, dx = \int_0^1 (1 - 2t)(-1) - (t^2 + (1 - t)^2)(-1) dt = 2 \int_0^1 t^2 dt = \frac{2}{3}.
\]

To parametrize \( C_B \) we set \( r(t) = (-1 + 2t, 0) \) for \( 0 \leq t \leq 1 \). Then \( dy = 0 \) and \( dx = 2dt \). We find

\[
F_B = \oint_{C_B} M \, dy - N \, dx = \int_0^1 (-1 + 2t)^2 \cdot 2 dt = 2 \int_0^1 -1 + 4t - 4t^2 dt = -\frac{2}{3}.
\]

The desired flux is the sum of \( F_R, F_L, \) and \( F_B, \) which means that it is \( 5/3 \).

We could certainly apply Green’s Theorem, to find a different computation of the same flux. Let \( R \) be the triangle with the given vertices. Then the flux is given by

\[
\oint M \, dy - N \, dx = \int \int_R \text{div} \, F \, dA = \int_0^1 \int_{1-y}^{1+y} (1 + 2y) \, dx \, dy = \frac{5}{3}.
\]

(15) 4. Apply the component test to decide whether \( F(x, y, z) = e^{y+z^2}(1, x, 2x) \) is conservative, and if so, find the potential.

Compare with Problem 9 from Section 15.3. Set \( F = (M, N, P) \). Then

\[
M_y = e^{y+z^2} = N_x, \quad M_z = 2e^{y+z^2} = P_x, \quad N_z = 2xe^{y+z^2} = P_y.
\]
The component test tells us that \( F \) is conservative. By inspection, we see that 
\[
 f(x, y, z) = xe^{y+2x} + c
\] is a potential for \( F \).

5. Compute the curve integral \( \int_C F \, dr \) where 
\[
 F(x, y, z) = \frac{1}{x^2+y^2+z^2} (2x, 2y, 2z)
\] and the curve \( C \) is parametrized by 
\[
 r(t) = (-1 + \cos t, 1 + \sin t, \sin t + \cos t) \quad \text{for} \quad 0 \leq t \leq \pi/2.
\] This problem borrows from Problem 22 in Section 15.3. Note that the vector field is conservative, in fact 
\[
 f(x, y, z) = \ln(x^2 + y^2 + z^2)
\] is a potential. The curve goes from 
\[
 r(0) = (0, 1, 1) \quad \text{to} \quad r(\pi/2) = (-1, 2, 1).
\] Then
\[
 \int_C F \, dr = f(r(\pi/2)) - f(r(0)) = \ln 6 - \ln 2 = \ln 3.
\]

6. Calculate \( \oint_C (6y + x) \, dx + (y + 2x) \, dy \) where \( C \) is the curve defined by 
\[
 (x - 2)^2 + (y - 3)^2 = 4.
\] This is Problem 19 from Section 15.4. We apply Gauss' Theorem. Setting 
\[
 M = 6y + x \quad \text{and} \quad N = y + 2x,
\] we find that \( N_x = 2 \) and \( M_y = 6 \). We denote the disk inclosed in the circle \( C \) by \( R \). The theorem tells us that
\[
 \oint_C M \, dx + N \, dy = \int \int_R (N_x - M_y) \, dx \, dy = \int \int_R -4 \, dx \, dy = -16\pi.
\]

7. Find the area of the part of the plane \( 2x + 6y + 3z = 6 \) that is contained in the first octant, \( x, y, z \geq 0 \). Call this surface \( S \). To earn any points you **must** treat \( S \) as an implicitly defined surface and use the appropriate formula to compute its area.

Compare this with Problem 52 in Section 15.5. The implicit equation for the surface is 
\[
 F(x, y, z) = 2x + 6y + 3z - 6 = 0,
\] and \( \nabla F = (2, 6, 3) \). Set \( p = (0, 0, 1) \), then \( \nabla F \cdot p = 3 \neq 0 \). The shaddow of \( S \) in the \( xy \)-plane is the triangle \( R \) with vertices \((0, 0), (3, 0), \) and \((0, 1)\), which is of area \( 3/2 \), and \( p \) is perpendicular to the \( xy \)-plane. We find that 
\[
 |\nabla F| = \sqrt{2^2 + 6^2 + 3^2} = 7 \quad \text{and} \quad |\nabla F \cdot k| = 3.
\] The surface area of \( S \) is
\[
 \int \int_R \frac{|\nabla F|}{|\nabla F \cdot p|} \, dA = \frac{7}{2}.
\]