Throughout the exam, $\Omega$ is a solid that is bounded from below by the cone $z = 3\sqrt{x^2 + y^2}$ and from above by the paraboloid $x^2 + y^2 = 4 - z$. Suppose that its density is $\delta(x, y, z) = 5 - z$. We use the vector field $F(x, y, z) = (x, x + y, 5xz^2)$. The intersection of the cone and paraboloid, $x^2 + y^2 = 1$ and $z = 3$, is a curve and will be denoted by $C$.

(15) 1. Compute the volume ($\text{Vol}$) of the solid.

The cone and paraboloid intersect at height $z = 3$, i.e., when $\rho = \sqrt{x^2 + y^2} = 1$. We use cylindrical coordinates $(\rho, \theta, z)$ and we are within the solid if $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$, and $3 \leq z \leq 4 - \rho^2$:

$$\text{Vol}(\Omega) = \int_0^{2\pi} \int_0^1 \int_{3\rho}^{4-\rho^2} \rho \, dz \, d\rho \, d\theta$$

$$= 2\pi \int_0^1 \rho(4 - \rho^2 - 3\rho) \, d\rho$$

$$= \frac{3\pi}{2}$$

(15) 2. Compute the mass ($M$) of the solid.

We integrate the density over the solid.

$$M = \int_0^{2\pi} \int_0^1 \int_{3\rho}^{4-\rho^2} (5 - z)\rho \, dz \, d\rho \, d\theta$$

$$= 2\pi \int_0^1 \left( 5z - \frac{z^2}{2} \right) \bigg|_{z=3\rho}^{z=4-\rho^2} \rho \, d\rho$$

$$= 2\pi \int_0^1 (12\rho - 15\rho^2 + \frac{7}{2}\rho^3 - \frac{1}{2}\rho^5) \, d\rho$$

$$= \frac{43\pi}{12}$$
3. What is the average density of the solid?

The average density is the ratio of mass to volume, so it is $43/18$.

4. What is the $z$–coordinate of the centroid of $\Omega$? This is not the center of mass.

We calculate the first moment $M_{xy}$ with constant density 1:

$$
M_{xy} = \int_0^{2\pi} \int_0^1 \int_{3\rho}^{4-\rho^2} z\rho \, dz \, d\rho \, d\theta
$$

$$
= 2\pi \int_0^1 \left[ \frac{1}{2} z^2 \right]_{z=3\rho}^{z=4-\rho^2} \rho \, d\rho
$$

$$
= \pi \int_0^1 \rho (16 - 17\rho^2 + \rho^4) \, d\rho
$$

$$
= \frac{47\pi}{12}
$$

The $z$–coordinate of the centroid of $\Omega$ will be $\bar{z} = M_{xy}/\text{Vol} = \frac{47}{18}$.

5. Find the second moment $I_z$, for a rotation about the $z$–axis.

We calculate

$$
I_z = \int_0^{2\pi} \int_0^1 \int_{3\rho}^{4-\rho^2} \rho^2 \delta \rho \, dz \, d\rho \, d\theta
$$

$$
= 2\pi \int_0^1 \int_{3\rho}^{4-\rho^2} \rho^3(5 - z) \, dz \, d\rho
$$

$$
= 2\pi \int_0^1 \rho^3(12 - 15\rho + \frac{7}{2}\rho^2 - \frac{1}{2}\rho^4) \, d\rho
$$

$$
= \frac{25\pi}{24}
$$

6. Find the second moment $I$ for a rotation about an axis parallel to the $z$–axis through the point $(3, 4)$ in the $xy$–plane.

We apply the parallel axis theorem, $I = I_z + Mh^2$ where $h$ is the distance between the $z$–axis and the axis of rotation. With $h = \sqrt{3^2 + 4^2} = 5$, $M = \frac{43\pi}{18}$, and $I_z = \frac{25\pi}{24}$ we find $I = \frac{725\pi}{8}$.

7. Parameterize the cone shaped part of the surface of the solid, and compute the surface area.

We use polar coordinates in the shadow of the cone:

$$
r(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, 3\rho)
$$

with $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. We compute

$$
r_\theta = (-\rho \sin \theta, \rho \cos \theta, 0)
$$

$$
r_\rho = (\cos \theta, \sin \theta, 3)
$$

$$
r_\theta \times r_\rho = (3\rho \cos \theta, 3\rho \sin \theta, \rho)
$$

$$
|r_\theta \times r_\rho| = \rho \sqrt{10}
$$

As surface area ($AC$) we find

$$
AC = \int_0^{2\pi} \int_0^1 |r_\theta \times r_\rho| \, d\rho \, d\theta = 2\pi \sqrt{10} \int_0^1 \rho \, d\rho = \pi \sqrt{10}.
$$
8. The parabolic part of the surface \((AP)\) is the solutions of the equation \(H(x, y, z) = x^2 + y^2 + z - 4 = 0\) for \(3 \leq z \leq 4\). Use this implicit description to compute \(AP\).

With \(k\) the unit vector in the \(z\)-direction, we compute

\[
\nabla H = (2x, 2y, 1) \\
|\nabla H| = \sqrt{4(x^2 + y^2) + 1} \\
|\nabla H \cdot k| = 1
\]

For the parabolic part of the surface area we find

\[
AP = \int \int \frac{|\nabla H|}{|\nabla H \cdot k|} dA = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} r \, dr \, d\theta = \frac{\pi}{6} [5^{3/2} - 1]
\]

9. Find \(\text{curl} \, F\) for the vector field \(F(x, y, z) = (x, x + y, 5xz^2)\).

According to the standard formula we find \(\text{curl} \, F(x, y, z) = (0, -5z^2, 1)\).

10. Is \(F\) conservative?

No, as \(\text{curl} \, F(x, y, z) \neq 0\) we see that \(F\) is not conservative.

11. Find \(\text{div} \, F\).

According to the standard formula we find \(\text{div} \, F = 1 + 1 + 10xz = 2(1 + 5xz)\).

12. Find the outward flow \((OF)\) of \(F\) across the surface of the solid.

We use the Divergence Theorem and compute (the triple integral is over the solid, and the double integral is over its surface):

\[
\int \int F \cdot n \, dS = \int \int \int \text{div} \, F \, dV = 2 \int \int \int (1 + 5xz) \, dx \, dy \, dz
\]

Integration of \(5xz\) will give zero due to reasons of symmetry, so that \(OF\) is twice the volume of the solid, \(OF = 3\pi\).

13. Find the circulation of \(F\) along \(C\) in counter clockwise direction. The curve \(C\) is the intersection of the cone and paraboloid and it was mentioned earlier.

We apply Stokes’ Theorem,

\[
\oint_C F \cdot ds = \int \int (\nabla \times F) \cdot n \, d\sigma = \int \int (0, -5z^2, 1) \cdot (0, 0, 1) \, d\sigma = \int \int 1 \, d\sigma = \pi.
\]

The first equal sign comes from Stokes’ Theorem. We calculated the curl of \(F\) previously and we are using it. We are free to pick a surface that has \(C\) as bounding curve, and we pick a horizontal disk. It is of radius 1 and has an area of \(\pi\). Its normal vector is \((0, 0, 1)\), which explains the remaining steps in the calculation.

14. State Green’s Theorem, make sure to spell out the assumptions and the conclusion.

Look it up in your book.