Linear Algebra (Math 311) – Test 1

Problem 1. [10 Points] Find the reduced row-echelon form of the matrix

\[ A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 3 & 3 & 8 \end{pmatrix}. \]

Problem 2. [12 Points] Suppose the given matrices are the augmented coefficient matrices of systems of equations \( Ax = b \), brought into reduced row-echelon form. Read off the answers \( x \).

\( (1) \quad A = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

\( (2) \quad B = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

\( (3) \quad C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \)

Problem 3. [12 Points] Invert the matrix

\[ B = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}. \]

Problem 4. [12 Points] Let \( C \) be a square matrix. What does it mean that the matrix is non-singular? Provide 5 additional properties that are equivalent to the assumption that \( C \) is non-singular.

Problem 5. [10 Points] Define the concept of an elementary row operation and an elementary matrix.

Problem 6. [10 Points] Prove that if the vectors \( v_1, \ldots, v_n \) are linearly dependent, then one of the vectors is a linear combination of the other vectors.

Problem 7. [12 Points] Let \( A \) and \( B \) be square matrices of the same size. Prove or disprove the assertion

\[ \text{trace}(AB) = \text{trace}(A) \text{trace}(B) \]
Problem 8. [10 Points] Decide whether the following set of vectors is linearly independent or dependent.

\[ \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} \right\} . \]

Problem 9. [12 Points] Decide whether the following set of polynomials spans the vector space \( P_2 \) consisting of all polynomials of degree at most 2:

\[ \{ t^2 + t + 1, t^2 - 1, t + 1 \} . \]