Linear Algebra (Math 311) – Test 2

Problem 1. [20 Points] Consider the matrix

\[ A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 0 & 3 & 7 \end{pmatrix}. \]

1. Find the nullspace of \( A \) and its dimension.

2. Find a basis of the row space of \( A \) and its dimension.

3. Find a basis of the column space of \( A \) and its dimension.

4. Find the orthogonal complement to the row space of \( A \) and its dimension.

Problem 2. [10 Points] Consider the basis \( \{1, 1 - t, 1 + t + t^2\} \) of the space \( P_2 \) of polynomials of degree at most two. Find the coordinate vector of \( t - t^2 \) with respect to \( S \).

Problem 3. [5 Points] State the Cauchy–Schwarz inequality.

Problem 4. [30 Points] Consider the subspace \( W \) of \( \mathbb{R}^4 \) spanned by the set \( S \) and a vector \( v \):

\[ S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad v = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}. \]

1. Find the cosine of the angle between the first two vectors in \( S \).

2. Find the length of the third vector in \( S \).

3. Find the matrix of the standard inner product on \( W \) (obtained by restriction of the dot product on \( \mathbb{R}^4 \)) with respect to the basis \( S \).

4. Orthonormalize \( S \) (Gram–Schmidt).

5. Find the projection of \( v \) onto \( W \).
Problem 5. [15 Points] If $V$ is a finite dimensional vector space and $W$ is a subspace, the $W$ is finite dimensional. Prove it.

Problem 6. [20 Points] Let $V$ be an inner product space and $W$ a subspace.

1. Show, for every $v \in V$, $||\text{proj}_W v|| \leq ||v||$.

2. Define $W^\perp$ and show that $W \cap W^\perp = \{0\}$. 